TROTTER-KATO TYPE CONVERGENCE FOR SEMIGROUPS

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ABSTRACT. In this paper, we establish the convergence of semigroups that are strongly continuous on $(0, \infty)$. By using Laplace transform theory, we show some properties of semigroups and the convergence result.

Mathematics Subject Classification : 47D05

Key words and phrases : Semigroup of operators, resolvent, Laplace transform

1. Introduction

The Laplace transform theory of Banach space valued functions plays an important role in the theory of semigroup of operators. It is known that if $A$ is a generator of a certain semigroup, its resolvent $R(\lambda, A)$ is related to the Laplace transform of the semigroup ([1], [4]).

In this paper we study the convergence of sequences of semigroups that are strongly continuous on $(0, \infty)$ and satisfy the stability condition by Laplace transform theory. With the strong convergence of resolvents of approximation operators with not necessarily dense domain and the usual boundedness assumptions, we can establish strong convergence of semigroups that are strongly continuous on $(0, \infty)$.

In section 2, we give the basic definitions and theorems for semigroups that are strongly continuous on $(0, \infty)$. In section 3, we recall the Laplace-Stieltjes transform and its convergence theorem for our result. And we present our convergence result for semigroups that are strongly continuous on $(0, \infty)$. Then we introduce a sequence $\{X_n\}$ of Banach spaces approximating a Banach space $X$, and we establish the approximation of discrete sequence of semigroups that are continuous on $(0, \infty)$.

This paper is supported by the NSRI of Seoul Women’s University, 2004.
Throughout this paper $X$ is a Banach space, all operators are linear and $M$, $\omega$ are constants. By $L(X)$, we denote the space of all bounded linear operators from $X$ to $X$. For an operator $A$, we will write $D(A)$, $\rho(A)$ and $R(\lambda, A)$ for the domain, the resolvent set and the resolvent of $A$, respectively.

2. Semigroups that are strongly continuous on $(0, \infty)$

A family $\{T(t) : t > 0\}$ of operators in $L(X)$ is a semigroup that is strongly continuous on $(0, \infty)$ if $T(t + s) = T(t)T(s)$ for $t, s > 0$ and for each $x \in X$ $T(t)x$ is strongly continuous on $(0, \infty)$.

If $\lim_{t \to 0^+} T(tx) = x$ for all $x \in X$, then $\{T(t) : t > 0\}$ is a $C_0$ semigroup.

Next, we define the generator of a semigroup that is strongly continuous on $(0, \infty)$ by Laplace transform (see [6]).

**Definition 2.1** Let $\{T(t) : t > 0\}$ be a semigroup that is strongly continuous and locally integrable on $(0, \infty)$ and satisfies

$$\left\| \int_0^t T(s) ds \right\| \leq Me^{\omega t} \text{ for } t > 0.$$ 

If there exists an operator $A : D(A) \subset X \to X$ such that $(\omega, \infty) \subset \rho(A)$ and

$$R(\lambda, A) = (\lambda - A)^{-1} = \int_0^\infty e^{-\lambda t}T(t)dt$$

for $\lambda > \omega$,

then $A$ is the generator of $\{T(t) : t > 0\}$.

In [1], Arendt showed that the Hille-Yosida operator is the generator of the bounded semigroup that is strongly continuous on $(0, \infty)$ if $X$ has the Radon-Nikodym property. In this case, the semigroup is given by the exponential formula (see [3]), that is, $T(t)x = \lim_{n \to \infty} (I - t/nA)^{-n}x$ for $x \in X$ and $t > 0$.

If $X$ is a Hilbert space, there is a generation theorem for a semigroup that is strongly continuous on $(0, \infty)$ (see [6]).

**Theorem 2.2** Let $A$ be the generator of a semigroup $\{T(t) : t > 0\}$ that is strongly continuous on $(0, \infty)$ and satisfies $\left\| \int_0^t T(s) ds \right\| \leq Me^{\omega t}$ for $t > 0$. Then

(i) For $x \in D(A)$ and $t > 0$, $T(tx) \in D(A)$, $T(t)Ax = AT(t)x$ and $T(t)x - x = \int_0^t T(s)Ax ds$.

(ii) For $x \in X$ and $t > 0$, $\int_0^t T(s)x ds \in D(A)$ and $T(t)x - x = A \int_0^t T(s)x ds$. 
**Proof.** By the definition of the generator \(A\), it is easy to show that (i) holds. Let \(x \in X\) and \(\lambda > \omega\). Then by integration by parts and the semigroup property,

\[
R(\lambda, A)(T(t)x - x) = \int_0^\infty e^{-\lambda s} T(s)(T(t)x - x)ds
\]

\[
= \int_0^\infty e^{-\lambda s}(T(s+t)x - T(s)x)ds
\]

\[
= \lambda \int_0^\infty e^{-\lambda s} \left( \int_t^{s+t} T(r)xdr - \int_0^s T(r)xdr \right) ds
\]

\[
= \lambda \int_0^\infty e^{-\lambda s} \left( \int_s^{s+t} T(r)xdr - \int_0^t T(r)xdr \right) ds
\]

\[
= \lambda \int_0^\infty e^{-\lambda s} T(s) \left( \int_0^t T(r)xdr \right) ds - \int_0^t T(r)xdr
\]

\[
= \lambda R(\lambda, A) \int_0^t T(r)xdr - \int_0^t T(r)xdr.
\]

Thus (ii) follows. \(\Box\)

Next, we consider the continuity of \(\{T(t) : t > 0\}\) at \(t = 0\). If \(T(t)x = 0\) for all \(t > 0\), then

\[
R(\lambda, A)x = \int_0^\infty e^{-\lambda t}T(t)xdt = 0.
\]

Since \(R(\lambda, A)\) is injective, \(x = 0\). So \(\{T(t) : t > 0\}\) is non-degenerate.

Suppose that \(\lim_{s \to 0^+} T(s)x = T(0)x\) exists. Then

\[
T(t)T(0)x = T(t) \lim_{s \to 0^+} T(s)x = \lim_{s \to 0^+} T(t+s)x = T(t)x.
\]

Thus \(T(0)x = x\), that is, if \(\{T(s) : s > 0\}\) is strongly continuous at \(s = 0\), then \(\lim_{s \to 0^+} T(s)x = x\).

**Theorem 2.3**  Let \(A\) be the generator of a semigroup \(\{T(t) : t > 0\}\) that is strongly continuous on \((0, \infty)\) and satisfies \(||T(t)|| \leq Me^{\omega t}\) for \(t > 0\). Let

\[
D = \{x \in X : \lim_{s \to 0^+} T(s)x = x\}.
\]

Then \(D = \overline{D(A)}\).

**Proof.** By theorem 2.2, \(\int_0^t T(s)xds \in D(A)\) and so \(T(t)x \in \overline{D(A)}\) for \(x \in X\). Let \(x \in D\). By the continuity of \(T(t)x\) at \(t = 0\), \(x \in \overline{D(A)}\).
To show that $\overline{D(A)} \subset D$, we will first show that $D$ is closed. Let $x \in D$ and $0 < t \leq 1$. Then there exist $x_n \in D$ such that $\lim_{n \to \infty} x_n = x$. So we have
\[
||T(t)x - x|| \leq ||T(t)x - T(t)x_n|| + ||T(t)x_n - x|| + ||x_n - x||
\]
\[
\leq (Me^\omega + 1)||x_n - x|| + ||T(t)x_n - x||.
\]
Thus $D$ is closed. For $x \in D(A)$, $T(t)x - x = \int_0^t T(s)Axds$ and so
\[
||T(t)x - x|| \leq Mte^{\omega t}||Ax||.
\]
Since $D$ is closed, we have $\overline{D(A)} \subset D$. □

Remark. Let $\{T(t) : t > 0\}$ be an exponentially bounded semigroup that is strongly continuous on $(0, \infty)$ with generator $A$. Then $\{T(t) : t > 0\}$ is a $C_0$ semigroup on $\overline{D(A)}$.

3. Convergence theorem

One tool we use to establish our result in this paper is the Laplace-Stieltjes transform and its convergence theorem. We recall some definitions and a known result for the Laplace-Stieltjes transform (see [2]).

Let $Lip_\omega$ be the space of all functions $F : [0, \infty) \to X$ with $F(0) = 0$ and
\[
||F(t) - F(s)|| \leq M \int_s^t e^{\omega r} dr \text{ for all } 0 \leq s \leq t.
\]
Then $Lip_\omega$ is a Banach space with norm
\[
||F||_{Lip_\omega} = \inf \left\{ M : ||F(t) - F(s)|| \leq M \int_s^t e^{\omega r} dr \text{ for all } 0 \leq s \leq t \right\}.
\]
For $F \in Lip_\omega$, the Laplace-Stieltjes transform of $F$ can be defined as $r(\lambda) = \int_0^\infty e^{-\lambda t} dF(t)$ for $\lambda > \omega$. The following convergence theorem is given in [2].

**Theorem 3.1** Let $F_n$, $F \in Lip_\omega$ and $||F||$, $||F_n|| \leq M$. Let $r_n$ and $r$ be the Laplace-Stieltjes transforms of $F_n$ and $F$, respectively. Then the following statements are equivalent.

(i) $\lim_{n \to \infty} r_n(\lambda) = r(\lambda)$ for $\lambda > \omega$.

(ii) $\lim_{n \to \infty} F_n(t) = F(t)$, uniformly on bounded $t$-intervals.
If $u_n, u : (0, \infty) \to X$ are locally integrable functions satisfying $\|u(t)\| \leq Me^{\omega t}$ for $t > 0$, then $F_n(t) = \int_0^t u(s)ds \in Lip_\omega$ and
\[
\int_0^\infty e^{-\lambda t}u_n(t)dt = \int_0^\infty e^{-\lambda t}dF_n(t).
\]
By theorem 3.1, the convergence of Laplace transforms of $u_n(t)$ to $u(t)$ is equivalent to the convergence of $\int_0^t u_n(s)ds$ to $\int_0^t u(s)ds$, uniformly on bounded $t$-intervals.

**Theorem 3.2** Let $\{T_n(t) : t > 0\}$ and $\{T(t) : t > 0\}$ be semigroups that are strongly continuous on $(0, \infty)$ and satisfy
\[
\|T_n(t)\|, \|T(t)\| \leq Me^{\omega t}
\]
for all $t > 0$ with generators $A_n$ and $A$, respectively. Let $D$ be a core for $A$. Consider the following statements.

(i) For each $x \in D$, there exist $x_n \in D(A_n)$ such that
\[
\lim_{n \to \infty} x_n = x \quad \text{and} \quad \lim_{n \to \infty} A_n x_n = Ax.
\]
(ii) $\lim_{n \to \infty} R(\lambda, A_n)x = R(\lambda, A)x$ for all $x \in X$ and $\lambda > \omega$.
(iii) $\lim_{n \to \infty} T_n(t)x = T(t)x$ for all $x \in D(A)$, uniformly on bounded $t$-intervals.

Then the implication (i) $\iff$ (ii) $\implies$ (iii) hold.

If $D(A)$ is dense, then the statements (i), (ii) and (iii) are equivalent.

**Proof.** The equivalence of (i) and (ii) is well known (e.g. see [5]).

(ii) $\implies$ (iii). Let $0 < t \leq T$ and $\lambda > \omega$. Since $\|T_n(t)\| \leq Me^{\omega t}$, it is enough to show that
\[
\lim_{n \to \infty} T_n(t)x = T(t)x
\]
for $x \in D(A)$.

Let $x \in D(A)$. Then $x = R(\lambda, A)y$ for some $y \in X$ and
\[
\|T_n(t)x - T(t)x\| = \|T_n(t)R(\lambda, A)y - T(t)R(\lambda, A)y\|
\leq \|T_n(t)R(\lambda, A)y - T_n(t)R(\lambda, A_n)y\|
+ \|T_n(t)R(\lambda, A_n)y - T(t)R(\lambda, A)y\|.
\]
By integration by parts and the semigroup property, the second term in the inequality becomes
\[
T_n(t)R(\lambda, A_n)y - T(t)R(\lambda, A)y \\
= \int_0^\infty e^{-\lambda s}T_n(t+s)yds - \int_0^\infty e^{-\lambda s}T(t+s)yds \\
= e^{\lambda t} \left( \int_t^\infty e^{-\lambda r}T_n(r)ydr - \int_t^\infty e^{-\lambda r}T(r)ydr \right) \\
= e^{\lambda t} (R(\lambda, A_n)y - R(\lambda, A)y) \\
- e^{\lambda t} \left( \int_0^t e^{-\lambda r}T_n(r)ydr - \int_0^t e^{-\lambda r}T(r)ydr \right) \\
= e^{\lambda t} (R(\lambda, A_n)y - R(\lambda, A)y) - \left( \int_0^t T_n(s)yds - \int_0^t T(s)yds \right) \\
- \lambda e^{\lambda t} \left( \int_0^t e^{-\lambda r} \left( \int_0^r T_n(s)yds - \int_0^r T(s)yds \right) dr \right).
\]
Thus we have
\[
||T_n(t)x - T(t)x|| \leq (Me^{\omega t} + e^{\lambda T}) ||R(\lambda, A_n)y - R(\lambda, A)y|| \\
+ \left\| \int_0^t T_n(s)yds - \int_0^t T(s)yds \right\| \\
+ \lambda e^{-\lambda t} \int_0^t e^{-\lambda r} \left\| \int_0^r T_n(s)yds - \int_0^r T(s)yds \right\| dr.
\]
By Theorem 3.1, the result follows.

Suppose that \( D(A) \) is dense in \( X \). Then \( \overline{D(A)} = X \) and
\[
R(\lambda, A_n)x - R(\lambda, A)x = \int_0^\infty e^{-\lambda t}(T_n(t)x - T(t)x)dt.
\]
By dominated convergence theorem, (iii) implies (ii).

Next, we consider the convergence of discrete convergence of semigroups. Let \( \{X_n\} \) be a sequence of Banach spaces with norm \( || \cdot ||_n \) approximating \( X \) in the following sense: For each \( n \) there exists bounded operator \( P_n : X \to X_n \) such that
\[
limit_{n \to \infty} ||P_n x||_n = ||x|| \quad \text{for all} \quad x \in X
\]
and there exists a constant \( C \), independent of \( n \), such that \( ||P_n||_n \leq C \) for all \( n \).

With the same argument of theorem 3.2, we can show the following approximation theorem.
Theorem 3.3 Let \( \{T_n(t) : t > 0\} \) be semigroups on \( X_n \) that are strongly continuous on \((0, \infty)\) and satisfy \( \|T_n(t)\| \leq Me^{\omega t} \) for all \( t > 0 \) with generators \( A_n \). Let \( \{T(t) : t > 0\} \) be a semigroup on \( X \) that is strongly continuous on \((0, \infty)\) and satisfies \( \|T(t)\| \leq Me^{\omega t} \) for all \( t > 0 \) with generator \( A \).

Suppose that
\[
\lim_{n \to \infty} ||R(\lambda, A_n)P_n x - P_n R(\lambda, A)x||_n = 0
\]
for all \( x \in X \). Then
\[
\lim_{n \to \infty} ||T_n(t)P_n x - P_n T(t)x||_n = 0
\]
for all \( x \in D(A) \).

References


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