ALTERNATIVE NUMERICAL APPROACHES TO THE JUMP-DIFFUSION OPTION VALUATION

BYUNGWOOK CHOI, HOSAM KI, MIYOUNG LEE*

Abstract. The purpose of this paper is to propose several approximating methods to obtain the American option prices under jump-diffusion processes. The first method is to extend an approximating method to the optimal exercise boundary by a multipiece exponential function suggested by Ju [17]. The second approach is to modify the analytical methods of MacMillan [20] and Zhang [25] in a discrete time space. The third approach is to apply the simulation technique of Ibáñez and Zapareto [14] to the problem of American option pricing when the jumps are allowed. Finally, we compare the numerical performance of each suggesting method with those of the previous numerical approaches.

AMS Mathematics Subject Classification : 60H20, 65C05, 65C30, 65Y20. Key words and phrases : American option pricing, jump-diffusion process, stochastic differential equation, optimal exercise boundary, multipiece exponential function, Monte-Carlo simulation.

1. Introduction

The value of an American option is greater than or equal to that of a European counterpart since there are more exercising opportunities in an American option than in European one. The value of an American call option on a non-dividend paying stock is equivalent to that of a European counterpart since it is known that exercising the American call option before the maturity date is never optimal. In general, the value of an American option might be decomposed in two terms: a European counterpart and an early exercise premium.
For the holder of an American option, the exercise time of an option is a crucial decision making problem. Intuitively, one might expect that the holder of an American option chooses his or her exercise policy in such a way that the expected payoff from the option is maximized. For example, in exercising an American call option written on a dividend paying stock, one should consider not only dividends from the stock that will be received until the maturity date, but also interests which otherwise can be earned on the exercise price. The optimal exercise of the American option can be characterized by optimal stopping time or optimal exercise boundary.

The optimal exercise boundary is a set of critical stock prices under which for an American put (or over which for an American call) it is optimal to exercise the option immediately. Since it is straightforward to compute an American option price, once the optimal exercise boundary is known, obtaining the critical stock prices efficiently is a possible strategy for pricing American options and deriving them with more efficient ways is the main objectives of this research.

One of the assumptions previously made in earlier works on the valuation of an option is that the underlying stock price follows a geometric Brownian motion as seen in Black-Scholes [3]. However, when the true return distribution of the underlying asset shows asymmetric leptokurtic features such as a high peak or heavier tails with left or right skewness, the option price may be mispriced. Another drawback is the so-called volatility smile: the implied volatility of an option as a function of its strike price resembles smile curve. To overcome these drawbacks, we will look at jump-diffusion processes as an alternative model introduced by Merton [21]. Under the jump-diffusion processes, the underlying asset is allowed to have jumps which imply that there is a positive probability of a stock-price change of some magnitude no matter how small the time interval between successive observations.

In the presence of jump processes, the valuation of derivatives is a complicating work since the market is incomplete. Merton [21] overcomes this difficulty by assuming that the jump component of the stock’s return is due to non-systematic risk, and therefore this component earns the risk-free rate of return under the Capital Asset Pricing Model. Furthermore, Merton [21] presents closed form solution for the European option prices, when the jump size is assumed to be a log-normal distribution. Another approach to handle a valuation problem in an incomplete market is based on the local risk minimizing strategy and using its associated minimal martingale measure discussed in several papers by Föllmer, Schweizer, and Sondermann (see [8], [9], [24]). Naik and Lee [22] and Ahn [1] use a general equilibrium framework to price options on the market portfolio with discontinuous returns.

Amin [2] provides a numerical scheme to price the European and American options by modifying the binomial tree model with allowing multiple jumps.
This method expresses the price of a derivative as a recursive function of its price at the previous time step and obtains the price with a backward dynamic programming. Zhang [26] applies the variational inequality techniques developed in Jaillet, Lamberton, and Lapeyre [16] for the American option pricing problem to the same problem but with jump-diffusion processes and proposes as the solution technique the semi-implicit finite difference method. Like Amin [2], her research presents numerical examples of American option prices under jump-diffusions.


In this paper, we propose three different numerical approaches to the valuation of American options under the jump-diffusion processes. The first approach is an extending approximating method to the optimal exercise boundary by a multipiece exponential function previously introduced by Ju [17]. The second approach is modifying analytical methods of MacMillan [20] and Zhang [25] for the valuation of American options under jump-diffusion processes. Finally we apply the simulation technique of Ibáñez and Zapareto [14] to the problem of American option pricing with jumps.

In general the Monte-Carlo simulation techniques in American option pricing are classified into two groups. The first approach computes directly the value function for option prices approximately (e.g., Broadie and Glasserman [4], and Longstaff and Schwartz [19]), and the second one calculates the optimal boundary of options before obtaining the option prices (e.g., Grant, Vora, and Weeks [12], and Ibáñez and Zapareto [14]). Please refer to the study of Fu, Laprise, Madan, Su and Wu [10] for further references in this research area.

The outline of the rest of this paper is as follows. In section 2 we make assumptions, present the necessary frameworks, and formulate problems. We begin in section 3 with a suggestion of new three approximating algorithms, which modify or extend the previous methods to this jump-diffusion problem. In section 4, we implement our algorithms as well as the previous approaches, compare the critical stock prices and option prices, and discuss the numerical efficiency of each method. Finally, section 5 concludes this paper.
2. Formulation of problems

In this section, we make some assumptions on a market, tradable assets, price processes of the assets under the risk-neutral measure, and formulate American option pricing problems under jump-diffusion processes.

We assume throughout this paper that (1) the capital markets are frictionless, and trading takes place continuously and without transaction costs, (2) there are two tradable assets in the market, a risky asset and a risk-free asset, (3) the short-term risk-free interest rate is known and constant through time, and (4) the stock price follows a jump-diffusion process through time. The risk-free asset $S_0$ at time $t$ is governed by the equations $dS_0^t = rS_0^t dt$, and $S_0^0 = 1$.

We consider in this paper an American put option written on a non-dividend paying stock in the presence of jumps. Due to the jump part, the market is incomplete and so there are infinite numbers of equivalent martingale measures in this market. Several approaches are introduced to handle and simplify the valuation problem of options in an incomplete markets. Either following Merton’s assumption [21] or adopting minimal martingale measure proposed by Föllmer, Schweizer, and Sondermann (see [8], [9], [24]) gives the following stochastic differential equation as dynamics of the stock price, $S_t$, under the risk-neutral measure:

$$\frac{dS_t}{S_t} = (r - \lambda k)dt + \sigma dW_t + d\left(\sum_{j=1}^{N_t} U_j\right).$$ (1)

To be more rigorous, we consider a probability space $(\Omega, \mathcal{F}, P)$ with $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}$, a filtration satisfying the usual conditions on which we define a standard Brownian motion $(W_t)_{t\geq 0}$, a Poisson process $(N_t)_{t\geq 0}$ with jump intensity $\lambda$ (the average number of arrivals per unit time) and a sequence of $(U_j)_{j\geq 1}$ of independent, identically distributed random variables taking values in $(-1, +\infty)$. In the above equation, $k \equiv \mathbb{E}U_1$, where $\mathbb{E}$ denotes the expectation operator under the risk-neutral measure $P$. We assume that the $\sigma$-algebras generated respectively by $(W_t)_{t\geq 0}, (N_t)_{t\geq 0}, (U_j)_{j\geq 1}$ are independent. For simplicity, we take the risk-free interest rate $r$ and the volatility $\sigma$ to be a constant and assume that the asset pays no dividend. Then the dynamics of $(S_t)_{t\geq 0}$ is given by

$$S_t = S_0 \exp\left((r - \lambda k - \frac{\sigma^2}{2})t + \sigma W_t\right) \prod_{j=1}^{N_t} (1 + U_j),$$ (2)

where $\prod_{j=1}^{0} = 1$.

Now, consider an American option with maturity $T$, allowing a payoff $f(S_t)$ when it is exercised at time $t$. Then the value of an American option at time $t$,
Jump-Diffusion valuation

\[ v(S_t, t) \text{ is given by} \]
\[ v(x, t) = \sup_{\tau \in \mathcal{F}_{t,T}} \mathbb{E}(e^{-r(\tau-t)} f(S^x_{\tau,t})) , \]  

(3)

where \( \mathcal{F}_{t,T} \) is the set of all stopping times with values in \([t, T]\) and \((S^x_{u,t})\) is defined by:

\[ S^x_{u,t} = x \exp \left\{ (r - \gamma k - \frac{\sigma^2}{2})(u-t) + \sigma (W_u - W_t) \right\} \prod_{j=N_t+1}^{N_u} (1 + U_j). \]  

(4)

If the American option is a call, the payoff function, \( f(S_t) = \max(S_t - K, 0) \), and if it is a put, \( f(S_t) = \max(K - S_t, 0) \), where \( K \) is the exercise price of the option. As mentioned previously, there are several numerical methods to handle the American option pricing problem when there are jumps. Amin [2] extends binomial tree model to this problem by allowing multiple jumps, and Zhang develops a semi-implicit finite difference method by extending the finite difference method for the pure diffusion case. In particular, when the jump size follows a log-normal distribution with mean \( \mu_J \), and standard deviation \( \sigma \), Choi [6] derives an analytical pricing formula for an American put at time \( t \),

\[ P(x, t) = p(x, t) + rK \int_{t}^{T} e^{-r(s-t)} \left[ \int_{0}^{B_s} \psi(S_s, s-t; x) dS_s \right] ds, \]  

(5)

where \( p(x, t) \) is a European option price given by:

\[
\begin{align*}
p(x, t) &= \sum_{m=0}^{\infty} \frac{e^{-\lambda(T-t)}(\lambda(T-t))^m}{m!} p_m(x, t), \\
p_m(x, t) &= K \exp\{-(r - \gamma k)(T-t) - n\gamma\} \Phi(-d'_2(x, K, T-t)) \\
&\quad - S \Phi(-d'_1(x, K, T-t)),
\end{align*}
\]

and

\[
\begin{align*}
\int_{0}^{B_s} \psi(S_s, s-t; x) dS_s &= \sum_{m=0}^{\infty} \frac{e^{-\lambda(s-t)}(\lambda(s-t))^m}{m!} \Phi(-d'_2(x, B_s, s-t)),
\end{align*}
\]

where

\[
-d'_1,2(x, K, T-t) = \frac{\ln(K/x) - (r - \gamma k + \frac{\nu_m^2}{2}(T-t) \pm \frac{1}{2} \nu_m^2)(T-t)}{\nu_m \sqrt{T-t}},
\]

and \( p(x, t) \) is the value of European option price, \( \psi(S_s, s-t; S_t) \) is the transition density function which denotes the probability density function of the stock price.
at time \( s, S_s \), given the stock price at time \( t \) is \( S_t \), \( \Phi(\cdot) \) is the cumulative standard normal distribution function,

\[
\lambda' \equiv \lambda(1 + k), \gamma \equiv \ln(1 + k) \text{and} \nu_m^2 \equiv \sigma^2 + m \delta^2/(s - t).
\]

We need to know the critical stock prices ahead before computing the American option prices if using Equation (5). When the current stock price is equal to the critical stock price, the value matching condition\(^1\) yields the following integral equation.

\[
K - B_t = p(B_t, t) + rK \int_t^T e^{-r(s-t)} \left[ \int_0^{B_s} \psi(S_s, s-t; B_t) \, dS_s \right] ds. \quad (6)
\]

Computing the critical stock price, \( (B_t)_{t \in [0,T]} \), from \( t = t_{N-1} \) to \( t = t_0 \) recursively after discretizing the time horizon into \( N \) equally-spaced intervals, \( t_0, t_1, \ldots, t_N \), is developed, but it is a very time-consuming work. Thus, in the next section, we suggest several approximating methods to compute option prices with more efficient ways.

3. Alternative approaches

In this section, we propose several numerical methods to compute critical stock prices and their associated option prices in a more efficient way. The basic ideas for the approaches we introduce here were previously suggested to obtain American option prices in the pure diffusion case, but we extend the ideas to more complicated jump-diffusion cases by modifying their methods for the first time in the literature, as we know.

3.1. Extended approximating method by a multipiece exponential function

We have already looked that the price of American put with time to maturity of \( T \) and current stock price of \( S_0, P(S_0, 0) \), which is given by

\[
P(S, 0) = p(S, 0) + rK \int_0^T e^{-ru} \left[ \int_0^{B_u} \psi(S_u, u; S) \, dS_u \right] du, \quad (7)
\]

where \( p(S, 0) \) is the European option price, \( B_u \) is the critical stock price at time \( u \), and the second term of the right hand side.

\(^1\)Value matching condition for the American option prices is given by:

\[
\lim_{S_t \to B_t} P(S_t, t) = K - B_t.
\]
We approximate the critical stock price, $B_t$, as an exponential function $B \exp(bt)$ for the interval $[t_1, t_2]$, as suggested by Ju [17]. Now we define the integral:

$$I(t_1, t_2, S, B, b) = \int_{t_1}^{t_2} e^{-rt} \left[ \int_0^{Be^{bt}} \psi(S_t, t; S) \, dS_t \right] \, dt,$$

where

$$\int_0^{Be^{bt}} \psi(S_t, t; x) \, dS_t = \sum_{m=0}^{\infty} \frac{e^{-\lambda t} (\lambda t)^m}{m!} \Phi \left( -d'_2(S, Be^{bt}, t) \right).$$

The notations are defined in the previous section. It is pointed out that the above integral can be expressed analytically as the sum of normal distribution functions if there is no jump. However in the presence of jumps, the integral part can be evaluated by using trapezoidal rule.

Now we define $P_1, P_2, P_3$, etc., as the approximate American put values corresponding to approximating the early exercise boundary as a one-piece exponential function($B_{11}e^{b_{11}t}$), a two-piece exponential function($B_{21}e^{b_{21}t}, B_{22}e^{b_{22}t}$), and a three-piece exponential function($B_{31}e^{b_{31}t}, B_{32}e^{b_{32}t}, B_{33}e^{b_{33}t}$), etc., then the $P_i$’s are given by using the decomposition formula in Equation (7):

$$P_1 = p(S, 0) + rKI(0, T, S, B_{11}, b_{11}),$$

$$P_2 = p(S, 0) + rKI(0, T/2, S, B_{22}, b_{22}) + KI(T/2, T, S, B_{21}, b_{21}),$$

$$P_3 = p(S, 0) + rKI(0, T/3, S, B_{33}, b_{33}) + KI(T/3, 2T/3, S, B_{32}, b_{32}) + KI(2T/3, T, S, B_{31}, b_{31}).$$

(9)

To determine $B'$s and $b'$s, we apply the value-matching, and high-contact conditions\(^2\), and solve the simultaneous equations using Newton-Raphson method. Given the computed prices, $P_1, P_2, P_3$, the three point Richardson scheme(Geske and Johnson [11]) gives a nice approximate American put price, $\hat{P}_A$ as follows:

$$\hat{P}_A = 4.5P_3 - 4P_2 + 0.5P_1$$

(10)

Now we show how to construct simultaneous equations when approximating the optimal exercise boundary as each type of multi-piece exponential function. First, consider the value $P_1$, which uses only one-piece exponential function.

---

\(^2\)High contact condition for American option prices is given by:

$$\lim_{S_t \to B_t} \frac{\partial P(S_t, t)}{\partial S_t} = -1.$$
Applying value-matching and high-contact condition at $t = 0$ would yield the following equations:

$$K - B_{11} = p(B_{11}, 0) + rKI(0, T, B_{11}, B_{11}, b_{11})$$
$$-1 = -N(-d_1(B_{11}, K, T)) + rKIS(0, T, B_{11}, B_{11}, b_{11}),$$

where $I_S$ denotes $\frac{\partial I}{\partial S}$. The simultaneous equations above is numerically solvable, since there are two unknown parameters($B_{11}, b_{11}$), and two equations. Next, consider the value $P_2$, which uses two-piece exponential function. In the same manner, the value-match and high-contact conditions at $t = T/2$ yield the equations:

$$K - B_{21}e^{b_{21}T/2} = p\left(B_{21}e^{b_{21}T/2}, T/2\right)$$
$$+ rKI\left(0, T/2, B_{21}e^{b_{21}T/2}, B_{21}e^{b_{21}T/2}, b_{21}\right)$$
$$-1 = -N\left(-d_1\left(B_{21}e^{b_{21}T/2}, K, T/2\right)\right)$$
$$+ rKIS\left(0, T/2, B_{21}e^{b_{21}T/2}, B_{21}e^{b_{21}T/2}, b_{21}\right).$$

The remaining part of exponential function can be obtained as follows by evaluating at $t = 0$:

$$K - B_{22} = p(B_{22}, 0)$$
$$+ rKI(0, T/2, B_{22}, B_{22}, b_{22}) + rKI(T/2, T, B_{22}, B_{21}, b_{21})$$
$$-1 = -N(-d_1(B_{22}, K, T))$$
$$+ rKIS(0, T/2, B_{22}, B_{22}, b_{22}) + rKIS(T/2, T, B_{22}, B_{21}, b_{21}).$$

In this two-piece exponential function case, the simultaneous equations are also solvable, since there are four unknown parameters, and four equations.

Similarly, the same technique applied to the value of $P_3$ gives each pair of simultaneous equations at times $t = 2T/3$, $T/3$, 0, respectively:

(i) at $t = 2T/3$:

$$K - B_{31}e^{b_{31}2T/3} = p\left(B_{31}e^{b_{31}2T/3}, 2T/3\right)$$
$$+ rKI\left(0, T/3, B_{31}e^{b_{31}2T/3}, B_{31}e^{b_{31}2T/3}, b_{31}\right)$$
$$-1 = -N\left(-d_1\left(B_{31}e^{b_{31}2T/3}, K, 2T/3\right)\right)$$
$$+ rKIS\left(0, T/3, T, B_{31}e^{b_{31}2T/3}, B_{31}e^{b_{31}2T/3}, b_{31}\right).$$
(ii) at $t = T/3$:

$$\begin{align*}
K - B_{32}e^{b_{32}T/3} &= p\left(B_{32}e^{b_{32}T/3}, T/3\right) \\
&
+ rKI\left(0, T/3, B_{32}e^{b_{32}T/3}, B_{32}e^{b_{32}T/3}, b_{32}\right) \\
&
+ rKI\left(T/3, 2T/3, B_{32}e^{b_{32}T/3}, B_{31}e^{b_{31}T/3}, b_{31}\right)
\end{align*}$$

$$-1 = -N\left(-d_1\left(B_{32}e^{b_{32}T/3}, K, 2T/3\right)\right)$$

$$\begin{align*}
&
+ rKI_S\left(0, T/3, B_{32}e^{b_{32}T/3}, B_{32}e^{b_{32}T/3}, b_{32}\right) \\
&
+ rKI_S\left(T/3, 2T/3, B_{32}e^{b_{32}T/3}, B_{31}e^{b_{31}T/3}, b_{31}\right).
\end{align*}$$

(iii) at $t = 0$:

$$\begin{align*}
K - B_{33} &= p(B_{33}, 0) \\
&
+ rKI(0, T/3, B_{33}, B_{33}, b_{33}) + rKI(T/3, 2T/3, B_{33}, B_{32}, b_{32}) \\
&
+ rKI(2T/3, T, B_{33}, B_{31}, b_{31}) \\
-1 &= -N\left(-d_1\left(B_{33}, K, T\right)\right) \\
&
+ rKI_S(0, T/3, B_{33}, B_{33}, b_{33}) + rKI_S(T/3, 2T/3, B_{33}, B_{33}, b_{32}) \\
&
+ rKI_S(2T/3, T, B_{33}, B_{31}, b_{31}).
\end{align*}$$

For each set of equation, one can use two-dimensional Newton-Raphson method to compute the unknown parameters ($B$'s and $b$'s). Once the unknown parameters are found, the prices of American puts can be easily obtained from Equations (8),(9), and (10). The numerical examples for the American option prices using this method are represented in Tables 1 and 2.

### 3.2. Analytical method

Another approach to obtaining the critical stock prices is based on the quadratic approximating approach orginally suggested by MacMillan [20] and extend by Zhang [25] to American option pricing under jump-diffusion processes. The proposed method here use only one part of their algorithm, which calculates the critical stock prices approximately based on the fixed point theorem. However the method we propose in this paper is different from theirs in that we compute the critical stock prices at every discrete times, $t_0, t_1, \ldots, t_N$, by dividing the time to maturity $T$ in $N$ equally-space intervals. Given the critical stock prices, $(B_{tn})_{n=0,1,\ldots,N}$, we can compute the American option prices using the decomposition formula of Equation (5) instead of using less efficient approximating pricing formula of Zhang [25].
The critical stock prices at time \( t_i \) for \( i \in \{0, 1, \ldots, N\} \) in the jump-diffusion case are obtained by using the first two steps, and option prices are computed in the final step below.

- **Step I:** Compute negative value of \( \eta \) satisfying \( \phi(\eta) = 0 \), where:

\[
\phi(\alpha) = \frac{\sigma^2}{2} \alpha^2 + \left( r - \lambda k - \frac{\sigma^2}{2} \right) \alpha - \left( r + \lambda + \frac{1}{T - t_i} \right) + \lambda \exp \left\{ \frac{\alpha^2 \delta^2}{2} + \alpha m \right\},
\]

for \( i = 1, 2, \ldots, N \).

- **Step II:** Compute the critical stock price \( B_{t_i} \) satisfying \( f(B_{t_i}) = B_{t_i} \), where

\[
f(x) = |\eta| \frac{K - p(x, t_i)}{p(x, t_i) + 1 + |\eta|},
\]

where \( p(x, t_i) \) is a European option price, and \( p'(x, t_i) = \frac{\partial p}{\partial x} \).

- **Step III:** Calculate the option prices using Equation (5), given the critical stock prices, \( (B_{t_n})_{n=0,1,\ldots,N} \) estimated in the Step II.

The parameters such as \( \sigma, r, \lambda, \delta, m \) are defined in the previous section. The numerical results for the American option prices using this scheme are shown in Tables 1 and 2. The approach we propose here improves the numerical efficiency prominently by reducing pricing errors ten times more than the original Zhang’s method as we see in Table 1.

### 3.3. Simulation method

In order to obtain the price of an American option, we compute the optimal exercise boundary by applying the simulation technique proposed by Ibáñez and Zaparote [14] to the jump-diffusion case. As mentioned previously, the optimal exercise boundary is the level of the stock price under which the value of exercise is equal to that of holding the option. In the words, the optimal exercise boundary, \( \{B_t; t \in [0, T]\} \) satisfies the following value-matching condition:

\[
V(B_t) = f(B_t) \equiv \begin{cases} 
B_t - K, & \text{for a call option} \\
K - B_t, & \text{for a put option}, 
\end{cases}
\]

where \( V(\cdot) \) is the value of an American option, and \( f(\cdot) \) is the payoff function at the time of exercise. As seen in the above equation (19), before computing the critical stock price at time \( t \), we need to know the critical stock prices at future time \( s(t < s < T) \) in advance recursively.
Given the initial time $t$, if the option matures at $T$, we divide the time horizon $T - t$ in $N$ equally-spaced intervals, with exercise dates $t_1, t_2, \ldots, t_N$ and $t_N = T$. We start to compute the optimal boundary from the time $t_{N-1}$, and compute the remaining boundary recursively.

We let $S^n_i$ denote the stock price at time $t_n$ in the $i$-th iteration, and $B^n_i$ denote the critical stock price at time $t_n$. The following is the main steps of the ALGORITHM.

Step I:
In this step, we compute the optimal exercise boundary at the exercise time, $t = t_{N-1}$. Since the only exercise opportunity at the time $t = t_{N-1}$ is the maturity date, this option is of European style, and so find $B_{N-1}$ such that $p(B_{N-1}) = f(B_{N-1})$.

Step II:
In this step, we compute the optimal exercise boundary at the exercise time, $t = t_{N-2}$. First, select $B_{N-1}$ as the initial point of $S^i_{N-2}$. Next, compute $V(S^i_{N-2})$, which is given by

$$V(S^i_{N-2}) = E(e^{-r(t^* - t_n)}f(S_{t^*})|S^i_n),$$

where $t^* \in \{t_{n+1}, t_{n+2}, \ldots, T\}$ is the optimal stopping time defined as the first time such that $S^n_{i+j} \leq B^n_{i+j}$. In simulating random processes of $S^n_{i+j}$ when computing $V(S^i_n)$, use the equation for $j = 1, 2, \ldots, T - n$:

$$S^n_{i+j} = S^n_{i+j-1} \exp \left\{ (r - \lambda k - \frac{\sigma^2}{2})\Delta t + \sigma W_{\Delta t} \right\} \prod_{j=1}^{N\Delta t} (1 + U_j),$$

where $\Delta t = T/N$. Next, find $S^i_{N-2}$ such that

$$V(S^i_{N-2}) = f(S^i_{N-2}).$$

Finally, we repeat the procedure until $|S^i_{N-2} - S^i_{N-2}| < \epsilon$ for some small number $\epsilon$, and let $B_{N-2} = S^i_{N-2}$.

Step III:
We repeat Step II at the remaining times $t = t_{N-3}, t_{N-4}, \ldots, t_1$.

We use $B_{n+1}$, obtained in the previous step, as the initial value of $S^1_n$.

Step IV:
Based on the critical stock prices, $B_n$ ($n=1,2,\ldots,N$), derived in the previous steps, we compute the American option prices using the equation (20), replacing $S^n_i$ with the current stock price.
The numerical results for the American option prices using the simulation method are provided in Tables 1 and 2.

4. Numerical results

In this section, we report numerical results to demonstrate the efficiency of the three methods we have proposed in this paper. We compare our methods with two different types of jump-diffusion models proposed previously. The first one is an 100-time step integral equation method (hereafter INTEG), and the second one is an analytical approximating method of MacMaillan and Zhang (hereafter MZ).

We implement the modified analytical method with 20 time steps (hereafter MMZ), the three-piece exponential function method (hereafter EXP3), and simulation method with 100 time steps (hereafter SIMUL). Since the numerical values derived from MMZ method is not largely affected by the number of time steps and the elapsed time of the MMZ is less than one second up to 20 time steps, we use 20 time steps for MMZ.

Even if the integral equation method is a numerical scheme, it provides more accurate option prices than any approximating method as the number of time steps increases. Thus we choose the method as our benchmark for true prices of American options under jump-diffusion model. We use the root of mean squared errors (RMSE) to measure the overall accuracy of three new methods relative to the integral equation method, and they are reported in Tables 1 and 2.

We also present numerical results of binomial tree model with $N = 10,000$, to assess the effects of jumps when pricing American options. The total variance\(^3\) in the jump-diffusion model is composed of each variance of diffusion and jump part. In numerical examples, we consider only the simple case with having $k = 0$ so that the total variance is given by:

$$\sigma_{total}^2 = \sigma^2 + \lambda(e^{\delta^2} - 1).$$

To compare the jump-diffusion model with the diffusion model, the total variance is used in implementing the binomial tree model.

Figure 1 illustrates the critical stock prices obtained through four methods: integral equation method with and without jump, simulation methods, and modified MacMillan-Zhang method. In computing the critical prices of the

\(^3\)We can decompose the total variance of stock price return into the variance of diffusion component and that of jump component:

$$\sigma_{total}^2 = \sigma^2 + \lambda\mathbb{E}U_j^2.$$

In particular, if $\ln(1+U_j)$ follows a normal distribution with mean $\mu_j$, and standard deviation $\delta$, then $1+k = \mathbb{E}(1+U_j) = e^{\delta^2/2+\mu_j}$, and $\mathbb{E}U_j^2 = \mathbb{E}(1+U_j)^2 - \mathbb{E}(1+2U_j) = (k+1)^2e^{\delta^2} - 2k - 1$. 


stock, we use as the parameter values exercise price ($K$) 50, interest rate ($r$) 6%, volatility ($\sigma$) 30%, jump volatility ($\delta$) 15%, and jump intensity ($\lambda$) 0.5.

Tables 1 and 2 provide American option prices using five jump-diffusion methods and one diffusion model with varying the levels of parameters such as stock price, exercise price, time to maturity, interest rate, volatility of diffusion, volatility of jump size, and jump intensity. In the bottom of each table, we report RMSE for the four methods: MZ, MMZ, EPX3, and SIMUL. In Table 1, the performance of SIMUL is the best, and in Table 2, EXP3 is the best. In both numerical results, the performance of the three new methods we propose here is better than that of the previous one, MZ. However, it should be pointed out that even though the results of SIMUL is more accurate than MZ, it is more time consuming than MZ.

Thus there is a trade-offs between the accuracy and speed. This is not a case for the other two methods (MMZ, EXP3), because they compute the option prices promptly. Therefore the new methods, MMZ and EXP3, are dominating the previous method MZ with regard to accuracy and speed.

5. Concluding remarks

We have proposed three numerical methods for pricing American options under jump-diffusion processes. The two approximating methods (modified MacMillan-Zhang method, and modified Ju method) show better performance than the previous approximating scheme. However the simulation method with jumps does not show better efficiency, even though it yields satisfactory option values. When the number of underlying assets are two or more, it is one of our future research to assess the performance of simulation method, since the simulation technique is known to be adequate for multiple underlying assets. Another part of our future works is to extend the numerical procedures developed here to the valuation of American options on futures, barrier options, quanto options, and currency options.
Appendix

This figure represents the critical stock prices of an American option with exercise price of 50, as a function of time to maturity. The critical stock price is defined as the level of stock price below which it is optimal to exercise the option. We compute them using four numerical methods for American option prices:

(i) integral equation method without jump (INTEG - no jump),
(ii) integral equation method with jumps (INTEG),
(iii) simulation method (SIMUL), and
(iv) modified MacMillan-Zhang method (MMZ).

Note that the first method computes the critical stock prices with no jump. The following parameter values are used:

\[ K = 50, \ r = 6\%, \ \sigma_{\text{total}} = 31.8\%, \ \sigma = 30\%, \ \delta = 15\%, \ \lambda = 0.5. \]
Table 1. Comparison of American put prices (I)

This table provides American put prices with varying stock prices, exercise prices, time to maturities, interest rates, volatilities, jump volatilities, and jump intensities. They are calculated in five different ways. (1) INTEG denotes the integral equation method, (2) MZ denotes the MacMillan-Zhang method, (3) MMZ denotes the modified MacMillan-Zhang method, (4) EXP3 denotes the three-piece exponential function method, and (5) SIMUL denotes the simulation method. Binomial denotes binomial tree model of Cox, Ross and Rubinstein [7], which computes American option prices under the pure diffusion model. RMSE measures the accuracy of the numerical results relative to the integral equation method (INTEG).

<table>
<thead>
<tr>
<th>$(S, K, T-t, r, \sigma^2, \delta^2, \lambda)$</th>
<th>(1) INTEG</th>
<th>(2) MZ</th>
<th>(3) MMZ</th>
<th>(4) EXP3</th>
<th>(5) SIMUL</th>
<th>Binomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>(40, 30, .25, .08, .05, .05, 5)</td>
<td>0.675</td>
<td>0.684</td>
<td>0.677</td>
<td>0.675</td>
<td>0.661</td>
<td>0.639</td>
</tr>
<tr>
<td>(40, 35, .25, .08, .05, .05, 5)</td>
<td>1.688</td>
<td>1.706</td>
<td>1.692</td>
<td>1.688</td>
<td>1.670</td>
<td>1.876</td>
</tr>
<tr>
<td>(40, 40, .25, .08, .05, .05, 5)</td>
<td>3.630</td>
<td>3.661</td>
<td>3.639</td>
<td>3.630</td>
<td>3.650</td>
<td>4.043</td>
</tr>
<tr>
<td>(40, 45, .25, .08, .05, .05, 5)</td>
<td>6.734</td>
<td>6.784</td>
<td>6.750</td>
<td>6.735</td>
<td>6.682</td>
<td>7.117</td>
</tr>
<tr>
<td>(40, 50, .25, .08, .05, .05, 5)</td>
<td>10.698</td>
<td>10.772</td>
<td>10.718</td>
<td>10.701</td>
<td>10.610</td>
<td>10.933</td>
</tr>
<tr>
<td>(40, 30, 1.0, .08, .05, .05, 5)</td>
<td>2.720</td>
<td>2.779</td>
<td>2.745</td>
<td>2.711</td>
<td>2.706</td>
<td>2.848</td>
</tr>
<tr>
<td>(40, 35, 1.0, .08, .05, .05, 5)</td>
<td>4.604</td>
<td>4.676</td>
<td>4.642</td>
<td>4.590</td>
<td>4.627</td>
<td>4.817</td>
</tr>
<tr>
<td>(40, 40, 1.0, .08, .05, .05, 5)</td>
<td>7.030</td>
<td>7.110</td>
<td>7.083</td>
<td>7.010</td>
<td>7.030</td>
<td>7.309</td>
</tr>
<tr>
<td>(40, 45, 1.0, .08, .05, .05, 5)</td>
<td>9.955</td>
<td>10.040</td>
<td>10.026</td>
<td>9.930</td>
<td>9.898</td>
<td>10.274</td>
</tr>
<tr>
<td>(40, 50, 1.0, .08, .05, .05, 5)</td>
<td>13.320</td>
<td>13.402</td>
<td>13.396</td>
<td>13.292</td>
<td>13.303</td>
<td>13.652</td>
</tr>
<tr>
<td>(45, 45, .25, .09, .004, .039, 9)</td>
<td>0.855</td>
<td>1.042</td>
<td>0.869</td>
<td>0.861</td>
<td>0.867</td>
<td>1.416</td>
</tr>
<tr>
<td>(50, 45, .25, .09, .004, .039, 9)</td>
<td>0.349</td>
<td>0.417</td>
<td>0.355</td>
<td>0.353</td>
<td>0.347</td>
<td>0.233</td>
</tr>
<tr>
<td>(55, 45, .25, .09, .004, .039, 9)</td>
<td>0.166</td>
<td>0.193</td>
<td>0.169</td>
<td>0.168</td>
<td>0.165</td>
<td>0.023</td>
</tr>
<tr>
<td>(45, 45, 1.0, .09, .004, .039, 9)</td>
<td>1.920</td>
<td>2.321</td>
<td>1.960</td>
<td>1.978</td>
<td>1.929</td>
<td>2.267</td>
</tr>
<tr>
<td>(50, 45, 1.0, .09, .004, .039, 9)</td>
<td>1.014</td>
<td>1.197</td>
<td>1.037</td>
<td>1.048</td>
<td>1.012</td>
<td>0.923</td>
</tr>
<tr>
<td>(55, 45, 1.0, .09, .004, .039, 9)</td>
<td>0.540</td>
<td>0.637</td>
<td>0.551</td>
<td>0.557</td>
<td>0.551</td>
<td>0.350</td>
</tr>
</tbody>
</table>

RMSE: 0.118, 0.013, 0.015, 0.010
Table 2. Comparison of American put prices (II)

This table provides American put prices with varying stock prices, exercise prices, time to maturities, interest rates, volatilities, jump volatilities, and jump intensities. They are calculated in five different ways. (1) INTEG denotes the integral equation method, (2) MZ denotes the MacMillan-Zhang method, (3) MMZ denotes the modified MacMillan-Zhang method, (4) EXP3 denotes the three-piece exponential function method, and (5) SIMUL denotes the simulation method. Binomial denotes binomial tree model of Cox, Ross and Rubinstein [7], which computes American option prices under the pure diffusion model. RMSE measures the accuracy of the numerical results relative to the integral equation method (INTEG).

<table>
<thead>
<tr>
<th>Jump-diffusion model</th>
<th>Diffusion model</th>
</tr>
</thead>
<tbody>
<tr>
<td>(S, K, T − t, r, σ, δ, λ)</td>
<td>(1) INTEG</td>
</tr>
<tr>
<td>(80, 100, .25, .6, .3, .15, 1)</td>
<td>20.059</td>
</tr>
<tr>
<td>(90, 100, .25, .6, .3, .15, 1)</td>
<td>11.676</td>
</tr>
<tr>
<td>(100, 100, .25, .6, .3, .15, 1)</td>
<td>5.920</td>
</tr>
<tr>
<td>(110, 100, .25, .6, .3, .15, 1)</td>
<td>2.654</td>
</tr>
<tr>
<td>(120, 100, .25, .6, .3, .15, 1)</td>
<td>1.096</td>
</tr>
<tr>
<td>(90, 100, 1.0, .6, .3, .15, 1)</td>
<td>15.467</td>
</tr>
<tr>
<td>(100, 100, 1.0, .6, .3, .15, 1)</td>
<td>10.792</td>
</tr>
<tr>
<td>(120, 100, 1.0, .6, .3, .15, 1)</td>
<td>7.429</td>
</tr>
<tr>
<td>(130, 100, 1.0, .6, .3, .15, 1)</td>
<td>5.064</td>
</tr>
<tr>
<td>RMSE</td>
<td>0.008</td>
</tr>
</tbody>
</table>
REFERENCES


Byungwook Choi received his BS from Seoul National University and Ph.D. at University of Texas at Austin under the direction of Patrick Jaillet. His research interests are mathematical finance, valuation of financial derivatives, and financial engineering.
Department of Business Administration, Kokuk University, Seoul 143-701, Korea
e-mail: bwchoi@konkuk.ac.kr

Hosam Ki received his BS and Ph.D. from Seoul National University. His research interests are numerical analysis and financial engineering.
Department of International Trade, Kokuk University, Seoul 143-701, Korea
e-mail: hosamki@konkuk.ac.kr

Miyoung Lee received her BS from Seoul National University and Ph.D. at Purdue University. Her research interests are knowledge management simulation based on data mining and modelling, and financial engineering.
Department of Management Information Systems, Kokuk University, Seoul 143-701, Korea
e-mail: yura@konkuk.ac.kr