FUZZY DIFFERENTIAL EQUATIONS
WITH NONLOCAL CONDITION

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Abstract. We shall prove the existence and uniqueness theorem of a solution to the nonlocal fuzzy differential equation using the contraction mapping principle.

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1. Introduction

In modeling, analyzing, and predicting behaviors of physical and natural phenomena, greater and greater emphasis has been placed upon fuzzy methods. This is due to combinations of complexity, two kinds of uncertainty-randomness and fuzziness, and ignorance which are present in the formulation of a great number of these problems. A large class of physically important problems is described by fuzzy differential equations.

The differential equation
\[ x'(t) = f(t, x(t)), \quad x(0) = x_0 \]
has a solution provided \( f \) is continuous and satisfies a Lipschitz condition by C. Corduneanu[2]. Byszewski[1] investigated the existence and uniqueness of mild, strong, and classical solutions of a nonlocal Cauchy problem for a semilinear evolution equation.

On the other hand, Kaleva[5] discussed the properties of differentiable fuzzy set valued mappings and gave the existence and uniqueness theorem for a solution of the fuzzy differential equations \( x'(t) = f(t, x(t)) \) when \( f \) satisfies the Lipschitz condition. And Feng[4] studied the existence and uniqueness of a solution, the continuity of the solution with respect to the initial value and the stability of fuzzy stochastic differential equations.

In this paper, we prove the existence and uniqueness theorem of a solution to the nonlocal fuzzy differential equation

\[ x'(t) = f(t, x(t)), \quad t \in I = [0, a], \]

\[ x(0) = g(t_1, t_2, \cdots, t_p, x(\cdot)) + x_0, \]

where

\[ 0 < t_1 < t_2 < \cdots < t_p \leq a, \quad f : I \times L_2 \to L_2 \]

is mean square (m.s. for short) continuous fuzzy mapping with respect to \( t \) which satisfies a generalized Lipschitz condition, \( g : I^p \times L_2 \to L_2 \) satisfies a generalized Lipschitz condition, and \( x_0 \in L_2 \). Hence

\[ L_2 = \{ X \mid X \text{ is a fuzzy random variable(f.r.v. for short)} \} \text{ with } E(\|X\|^2) < \infty \}([4]). \]

The symbol \( g(t_1, t_2, \cdots, t_p, x(\cdot)) \) is used in the sense that in the place of \( \cdot \) we can substitute only elements of the set \( \{t_1, t_2, \cdots, t_p\} \). For example, \( g(t_1, \cdots, t_p, x(\cdot)) \) can be defined by the formula

\[ g(t_1, \cdots, t_p, x(\cdot)) = c_1 x(t_1) + c_2 x(t_2) + \cdots + c_p x(t_p) \]

where \( c_i (i = 1, 2, \cdots, p) \) are given constants.

2. Preliminaries

The symbol \( P_C(R^n) \) denotes the family of all nonempty compact convex subsets of \( R^n \). Define the addition and scalar multiplication in \( P_C(R^n) \) as usual. Denote \( E^n = \{ u : R^n \to [0, 1] | u \text{ satisfies } (i) - (iv) \text{ below} \} \), where

(i) \( u \) is normal, i.e., there exists an \( x_0 \in R^n \) such that \( u(x_0) = 1 \);
(ii) \( u \) is fuzzy convex, i.e., \( u(rx + (1-r)y) \geq \min(u(x), u(y)), x, y \in R^n, r \in [0,1] \);
(iii) \( u \) is upper semicontinuous;
(iv) \( [u]^0 = \{ x \in R^n | u(x) > 0 \} \) is compact.

Let \( u, v \in E^n \), and set

\[ D(u, v) = \sup_{0 \leq r \leq 1} d([u]^r, [v]^r), \]

where \( [u]^r = \{ x \in R^n | u(x) \geq r \}, 0 < r \leq 1 \), is the \( r \)-level set of \( u \), \( d \) is the hausdorff metric defined in \( P_C(R^n) \). i.e.,

\[ d(A, B) = \max(\sup_{a \in A} \inf_{b \in B} |a - b|, \sup_{b \in B} \inf_{a \in A} |a - b|), \]
for all $A, B \in P_C(\mathbb{R}^n)$, where $| \cdot |$ denotes the usual Euclidean norm in $\mathbb{R}^n$. $(E^n, D)$ is a complete metric space [6].

Let $(\Omega, \mathcal{A}, P)$ be a complete probability space. A fuzzy random variable (f.r.v. for short) is a Borel measurable function $X : (\Omega, \mathcal{A}) \to (E^n, D)$. Let

$$L_2(\Omega, \mathcal{A}, P) = \{ X | X \text{ is a f.r.v. with } \int \Omega D(X, \hat{0})^2 dP(\omega) < \infty \}. $$

Two f.r.v.'s $X$ and $Y$ are called equivalent if $P(X \neq Y) = 0$. The all equivalent element in $L_2$ are identified. Define

$$\varphi(X, Y) = \left( \int \Omega (D(X, Y))^2 dP \right)^{\frac{1}{2}}, \quad X, Y \in L_2.$$ 

The norm $\|X\|_2$ of an element $X \in L_2$ is defined by

$$\|X\|_2 = \varphi(X, \hat{0}) = \left( \int \Omega (D(X, \hat{0}))^2 dP \right)^{\frac{1}{2}}.$$ 

Then $(L_2, \varphi)$ is a complete metric space [3, Corollary 2.2] and $\varphi$ satisfies that

$$\varphi(X + Z, Y + Z) = \varphi(X, Y), \quad \varphi(\lambda X, \lambda Y) = |\lambda| \varphi(X, Y),$$

$$\varphi(\lambda X, kX) \leq |\lambda - k|\|X\|_2$$

for any $X, Y, Z \in L_2$ and $\lambda, k \in \mathbb{R}$.

3. Nonlocal fuzzy differential equations

We consider the nonlocal fuzzy differential equation:

$$x'(t) = f(t, x(t)), \quad t \in I = [0, a],$$

(3.1)\[ x(0) = g(t_1, t_2, \cdots, t_p, x(\cdot)) + x_0 \]

where $0 < t_1 < t_2 \leq \cdots < t_p \leq a$, $f : I \times L_2 \to L_2$ is m.s. continuous fuzzy mapping with respect to $t$ which satisfies a generalized Lipschitz condition, $g : I^p \times L_2 \to L_2$ satisfies a generalized Lipschitz condition and $x_0 \in L_2$.

**Theorem 3.1.** Let $f : I \times L_2 \to L_2$ be m.s. continuous with respect to $t$ and there exists constants $L$ and $K$ such that

$$\varphi(f(t, x), f(t, y)) \leq L\varphi(x, y),$$
\[ \varphi(g(t_1, \cdots, t_p, x(\cdot)), g(t_1, \cdots, t_p, y(\cdot))) \leq K \varphi(x, y), \forall t \in I, x, y \in L_2. \]

Then the equation (3.1) has a unique solution on the interval \([0, \xi]\) where

\[ \xi = \min \left\{ a, \frac{b - N}{M}, \frac{1 - K}{L} \right\}, \varphi(f(t, x), \hat{0}) \leq M, \varphi(g(t_1, \cdots, t_p, x(\cdot)), \hat{0}) \leq N. \]

**Proof.** Let \( B = \{ x \in L_2 | H(x, x_0) \leq b \} \) be the space of m.s. continuous fuzzy mappings with

\[ H(x, y) = \sup_{0 \leq t \leq \xi} \varphi(x(t), y(t)) \]

and \( b \) a positive number. Define a mapping \( T : B \rightarrow B \) by

\[ T x(t) = x_0 + g(t_1, \cdots, t_p, x(\cdot)) + \int_0^t f(s, x(s))ds. \]

First of all, we show that \( T \) is m.s. continuous and \( H(T x, x_0) \leq b \). Since \( f \) is m.s. continuous, we have

\[ \varphi(T x(t + h), T x(t)) = \varphi \left( \int_0^{t+h} f(s, x(s))ds, \int_0^t f(s, x(s))ds \right) \]
\[ \leq \int_0^{t+h} \varphi(f(s, x(s)), \hat{0})ds \]
\[ \leq h M \rightarrow 0 \quad \text{(as \ } h \rightarrow 0). \]

That is, the map \( T \) is m.s. continuous on \( I \). Furthermore,

\[ \varphi(T x(t), x_0) \leq \varphi(g(t_1, \cdots, t_p, x(\cdot)), \hat{0}) + \varphi \left( \int_0^t f(s, x(s))ds, \hat{0} \right) \]
\[ \leq N + Mt, \]

and so

\[ H(T x, x_0) = \sup_{0 \leq t \leq \xi} \varphi(T x(t), x_0) \]
\[ \leq N + M \xi \]
\[ \leq b. \]

Since \((L_2, \varphi)\) is a complete metric space, a standard proof applies to show that

\[ C([0, \xi], L_2) = \{ x : [0, \xi] \rightarrow L_2 | x(t) \text{ is m.s. continuous} \} \]
is complete. Now we show that $B$ is a closed subset of $C([0, \xi], L_2)$. Let $\{x_n\}$ be a sequence in $B$ such that $x_n \to x \in C([0, \xi], L_2)$ as $n \to \infty$. Then
\begin{align*}
\varphi(x(t), x_0) &\leq \varphi(x(t), x_n(t)) + \varphi(x_n(t), x_0), \\
H(x, x_0) &= \sup_{0 \leq t \leq \xi} \varphi(x(t), x_0) \\
&\leq H(x, x_n) + H(x_n, x_0) \\
&\leq \varepsilon + b
\end{align*}
for sufficiently large $n$ and arbitrary $\varepsilon > 0$. So $x \in B$. This implies that $B$ is a closed subset of $C([0, \xi], L_2)$. Therefore $B$ is a complete metric space. Next, we will show that $T$ is a contraction mapping. For $x, y \in B$,
\begin{align*}
\varphi(Tx(t), Ty(t)) &\leq \varphi(g(t_1, \cdots, t_p, x(\cdot)), g(t_1, \cdots, t_p, y(\cdot))) \\
&\quad + \varphi \left( \int_0^t f(s, x(s))ds, \int_0^t f(s, y(s))ds \right) \\
&\leq K\varphi(x, y) + \int_0^t L\varphi(x(s), y(s))ds.
\end{align*}
Thus, we obtain
\begin{align*}
H(Tx, Ty) &\leq \sup_{0 \leq t \leq \xi} \left\{ K\varphi(x, y) + L \int_0^t \varphi(x(s), y(s))ds \right\} \\
&\leq (K + \xi L)H(x, y).
\end{align*}
Since $K + \xi L < 1$, $T$ is a contraction map. Therefore $T$ has a unique fixed point $Tx = x \in C([0, \xi], E^n)$, that is
\begin{equation}
x(t) = x_0 + g(t_1, \cdots, t_p, x(\cdot)) + \int_0^t f(s, x(s))ds.
\end{equation}
□

Theorem 3.2. Suppose that $f, g$ are the same as in Theorem 3.1. Let $x(t, x_0)$, $y(t, y_0)$ be solutions of Eq. (3.1) to $x_0$, $y_0$, respectively. Then there exist constants $c_1$ and $c_2$ such that
\begin{enumerate}
\item $(i) \quad H(x(\cdot, x_0), y(\cdot, y_0)) \leq c_1 \varphi(x_0, y_0)$ for any $x_0, y_0 \in L_2$,
\item $(ii) \quad H(x(\cdot, x_0), \hat{0}) \leq c_2(\varphi(x_0, \hat{0}) + N + M)$, where
\end{enumerate}
\begin{align*}
\varphi(g(t_1, \cdots, t_p, x(\cdot), \hat{0}) &\leq N \quad \text{and} \quad \int_0^t \varphi(f(s, \hat{0}), \hat{0})ds \leq M.
\end{align*}
Proof. (i) For any $t \in [0, \xi]$, we have
\[
\varphi(x(t, x_0), y(t, y_0)) \\
\leq \varphi(x_0 + g(t_1, \cdots, t_p, x(\cdot, x_0)) + \int_0^t f(s, x(s, x_0))ds, \\
y_0 + g(t_1, \cdots, t_p, y(\cdot, y_0)) + \int_0^t f(s, y(s, y_0))ds \\
\leq \varphi(x_0, y_0) + K \varphi(x(\cdot, x_0), y(\cdot, y_0)) \\
+ L \int_0^t \varphi(x(s, x_0), y(s, y_0))ds.
\]
From Gronwall’s inequality, we get
\[
\varphi(x(t, x_0), y(t, y_0)) \leq \left[ \varphi(x_0, y_0) + K \varphi(x(\cdot, x_0), y(\cdot, y_0)) \right] \exp L \xi.
\]
Thus we have
\[
H(x(\cdot, x_0), y(\cdot, y_0)) \leq \left[ \varphi(x_0, y_0) + K H(x(\cdot, x_0), y(\cdot, y_0)) \right] \exp L \xi.
\]
i.e.,
\[
(1 - K \exp L \xi) H(x(\cdot, x_0), y(\cdot, y_0)) \leq \varphi(x_0, y_0) \exp L \xi.
\]
Consequently, we obtain
\[
H(x(\cdot, x_0), y(\cdot, y_0)) \leq \frac{\exp L \xi}{1 - K \exp L \xi} \varphi(x_0, y_0).
\]
Taking $c_1 = \frac{\exp L \xi}{1 - K \exp L \xi}$, we obtain
\[
H(x(\cdot, x_0), y(\cdot, y_0)) \leq c_1 \varphi(x_0, y_0).
\]
(ii) For any $t \in [0, \xi]$,
\[
\varphi(x(t, x_0), \hat{0}) \leq \varphi(x_0, \hat{0}) + \varphi(g(t_1, \cdots, t_p, x(\cdot, x_0)), \hat{0}) \\
+ \int_0^t \varphi(f(s, x(s, x_0)), f(s, \hat{0}))ds \\
+ \int_0^t \varphi(f(s, \hat{0}), \hat{0})ds \\
\leq \varphi(x_0, \hat{0}) + \varphi(g(t_1, \cdots, t_p, x(\cdot, x_0)), \hat{0}) \\
+ L \int_0^t \varphi(x(s, x_0), \hat{0})ds + \int_0^t \varphi(f(s, \hat{0}), \hat{0})ds.
\]
From Gronwall’s inequality, we get
\[
\varphi(x(t, x_0), \hat{0}) \leq \left[ \varphi(x_0, \hat{0}) + \varphi(g(t_1, \cdots, t_p, x(\cdot), x_0)), \hat{0})
\right]
\]
\[
+ \int_0^t \varphi(f(s, 0), \hat{0}) ds \exp Lt
\]
\[
\leq (\varphi(x_0, \hat{0}) + N + M)\exp L\xi.
\]
Taking \(c_2 = \exp L\xi\), we get
\[
H(x(\cdot, x_0), \hat{0}) = \sup_{0 \leq t \leq \xi} \varphi(x(t, x_0), \hat{0})
\]
\[
\leq c_2(\varphi(x_0, \hat{0}) + N + M).
\]
\[
\square
\]

We consider the following fuzzy differential equations with nonlocal conditions:
\[
x(t) = x_0 + g(t_1, \cdots, t_p, x(\cdot)) + \int_0^t f(s, x(s)) ds,
\]
(3.2)
\[
x_n(t) = x_{n,0} + g_n(t_1, \cdots, t_p, x_n(\cdot)) + \int_0^t f_n(s, x_n(s)) ds, \quad n \geq 1.
\]
(3.3)

If Eqs. (3.2) and (3.3) satisfy the conditions of Theorem 3.1, then they have unique solutions \(x(t)\), and \(x_n(t), \quad t \in [0, \xi], \) respectively.

**Theorem 3.3.** Suppose that \(f, g\) are the same as in Theorem 3.1. If
\[
\varphi(x_{n,0}, x_0) \to 0,
\]
\[
\varphi(g_n(t_1, \cdots, t_p, x(\cdot)), g(t_1, \cdots, t_p, x(\cdot))) \to 0, \quad \text{and}
\]
\[
\sup_{0 \leq t \leq \xi} \varphi(f_n(t, y), f(t, y)) \to 0 \quad \text{as} \quad n \to \infty, \quad \text{for each} \quad y \in L^2,
\]
then
\[
\sup_{0 \leq t \leq \xi} \varphi(x_n(t), x(t)) \to 0 \quad \text{as} \quad n \to \infty.
\]

**Proof.** For any \(0 \leq t \leq \xi\), we have
\[
\varphi(x_n(t), x(t))
\]
\[
\leq \varphi(x_{n,0}, x_0) + \varphi(g_n(t_1, \cdots, t_p, x_n(\cdot)), g(t_1, \cdots, t_p, x(\cdot))
\]
\[
+ \int_0^t \varphi(f_n(s, x_n(s)), f(s, x(s))) ds
\]
\[ \leq \varphi(x_{n,0}, x_0) + \varphi(g_n(t_1, \ldots, t_p, x_n(\cdot)), g_n(t_1, \ldots, t_p, x(\cdot))) \\
+ \varphi(g_n(t_1, \ldots, t_p, x(\cdot)), g(t_1, \ldots, t_p, x(\cdot))) \\
+ \int_0^t \varphi(f_n(s, x_n(s)), f_n(s, x(s)))ds \\
+ \int_0^t \varphi(f_n(s, x(s)), f(s, x(s)))ds \\
\leq \varphi(x_{n,0}, x_0) + K\varphi(x_n(\cdot), x(\cdot)) \\
+ \varphi(g_n(t_1, \ldots, t_p, x(\cdot)), g(t_1, \ldots, t_p, x(\cdot))) \\
+ L \int_0^t \varphi(x_n(s), x(s))ds + \int_0^t \varphi(f_n(s, x(s)), f(s, x(s)))ds. \]

From Gronwall’s inequality, we get

\[ \varphi(x_n(t), x(t)) \leq \left[ \varphi(x_{n,0}, x_0) + K\varphi(x_n(\cdot), x(\cdot)) \\
+ \varphi(g_n(t_1, \ldots, t_p, x(\cdot)), g(t_1, \ldots, t_p, x(\cdot))) \\
+ \int_0^t \varphi(f_n(s, x(s)), f(s, x(s)))ds \right] \exp Lt. \]

That is,

\begin{align*}
(1-K\exp L\xi) \sup_{0 \leq t \leq \xi} \varphi(x_n(t), x(t)) \\
\leq \left[ \varphi(x_{n,0}, x_0) + \varphi(g_n(t_1, \ldots, t_p, x(\cdot)), g(t_1, \ldots, t_p, x(\cdot))) \\
+ \sup_{0 \leq t \leq \xi} \int_0^t \varphi(f_n(s, x(s)), f(s, x(s)))ds \right] \exp L\xi.
\end{align*}

(3.4)

And

\[ \varphi(f_n(s, x(s)), f(s, x(s))) \\
\leq \varphi(f_n(s, x(s)), f_n(s, \hat{0} )) + \varphi(f_n(s, \hat{0} ), f(s, \hat{0} )) \\
+ \varphi(f(s, \hat{0} ), f(s, x(s))) \\
\leq 2L\varphi(s, \hat{0}) + \sup_{0 \leq s \leq \xi} \varphi(f_n(s, \hat{0} ), f(s, \hat{0} )) \\
\leq 2Lc_2(\varphi(x_0, \hat{0}) + N + M) + 1 \]

as soon as \( n \) is large enough, where we used (ii) of Theorem 3.2. Hence, using the dominated convergence theorem in Eq. (3.4), we obtain the conclusion of the theorem. \( \square \)
REFERENCES


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