SKEW POLYNOMIAL RINGS OVER $\sigma$-QUASI-BAER AND $\sigma$-PRINCIPALLY QUASI-BAER RINGS

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Abstract. Let $R$ be a ring and $\sigma$ be an endomorphism of $R$. $R$ is called $\sigma$-rigid (resp. reduced) if $a\sigma(a) = 0$ (resp. $a^2 = 0$) for any $a \in R$ implies $a = 0$. An ideal $I$ of $R$ is called a $\sigma$-ideal if $\sigma(I) \subseteq I$. $R$ is called $\sigma$-quasi-Baer (resp. right (or left) $\sigma$-p.q.-Baer) if the right annihilator of every $\sigma$-ideal (resp. right (or left) principal $\sigma$-ideal) of $R$ is generated by an idempotent of $R$. In this paper, a skew polynomial ring $A = R[x; \sigma]$ of a ring $R$ is investigated as follows: For a $\sigma$-rigid ring $R$, (1) $R$ is $\sigma$-quasi-Baer if and only if $A$ is quasi-Baer if and only if $A$ is $\bar{\sigma}$-quasi-Baer for every extended $\sigma$-ideal of $A$; (2) $R$ is right $\sigma$-p.q.-Baer if and only if $A$ is right $\sigma$-p.q.-Baer if and only if $A$ is p.q.-Baer if and only if $A$ is $\bar{\sigma}$-p.q.-Baer for every extended endomorphism $\bar{\sigma}$ on $A$ of $\sigma$.

1. Introduction and some definitions

Throughout this paper, $R$ will denote an associative ring with identity, $\sigma$ will be an endomorphism of $R$, and $A$ will be the skew polynomial ring $R[x; \sigma]$, i.e., $A$ is a ring of polynomials over $R$ in an indeterminate $x$ with multiplication subject to the relation $xr = \sigma(r)x$ for all $r \in R$. When $\sigma$ is identity 1, we write $R[x]$ for $R[x; 1]$. In [11] Kaplansky introduced the Baer rings (i.e., rings in which the right annihilator of every nonempty subset is generated (as a right ideal) by an idempotent) to abstract various properties of rings of operators on Hilbert spaces. In [8], Clark introduced the quasi-Baer rings (i.e., rings in which the right annihilator of every right ideal is generated (as a right ideal) by an idempotent).
idempotent) which are generalizations of Baer rings and used them to characterize a finite dimensional twisted matrix units semigroup algebra over an algebraically closed field. Further works on quasi-Baer rings appear in [12], [3], [4] and [5]. The study of Baer and quasi-Baer rings has its roots in functional analysis. Recently, in [6] Birkenmeier, Kim and Park defined a right (or left) principally quasi-Baer (simply, called right (or left) p.q.-Baer) ring as a generalization of quasi-Baer ring by the rings in which the right (or left) annihilator of every right (or left) principal ideal of $R$ is generated by an idempotent of $R$. $R$ is called a p.q.-Baer ring if it is both right p.q.-Baer and left p.q.-Baer. Another generalization of Baer ring is a p.p.-ring. A ring $R$ is called a right (resp. left) p.p.-ring if the right (resp. left) annihilator of any element of $R$ is generated by an idempotent. We also define a right (or left) σ-p.q.-Baer (resp. right (or left) σ-p.p.-ring) by the ring in which the right (or left) annihilator of every right (or left) principal σ-ideal (resp. σ-element) is generated by an idempotent. $R$ is called a σ-p.q.-Baer ring (resp. σ-p.p.-ring) if it is both right and left p.p.-ring.

A subset $S$ of a ring $R$ is called a σ-set if $S$ is a σ-stable set, i.e., σ($S$) ⊆ $S$. In particular, if a singleton set $S = \{a\}$ of $R$ is σ-set, i.e., σ($a$) = $a$, then $a$ is called a σ-element of $R$. A left (right, two-sided) ideal $I$ of $R$ is called a left (right, two-sided) σ-ideal if $I$ is a σ-set. By analog, we can define a σ-Baer ring (resp. σ-quasi-Baer-ring) by the ring in which the right annihilator of every σ-set (resp. σ-ideal) is generated by an idempotent. We also define a right (or left) σ-p.q.-Baer ring (resp. right (or left) σ-p.p.-ring) by the ring in which the right (or left) annihilator of every right (or left) principal σ-ideal (resp. σ-element) is generated by an idempotent. $R$ is called a σ-p.q.-Baer ring (resp. σ-p.p.-ring) if it is both right σ-p.q.-Baer (resp. left σ-p.q.-Baer) and left σ-p.p.-Baer (resp. right σ-p.p.-Baer). In this paper, we denote the right (resp. left) annihilator of a subset $S$ of a ring $R$ by $r_R(S) = \{a \in R \mid Sa = 0\}$ (resp. $l_R(S) = \{a \in R \mid aS = 0\}$). We recall that $R$ is a σ-rigid (resp. reduced) ring if for some endomorphism σ of $R$, $\sigma(a) = 0$ (resp. $a^2 = 0$) implies that $a = 0$ for each $a \in R$. We can note that any σ-rigid ring is reduced and this endomorphism σ is a monomorphism. Now we can observe the following implications: Baer (resp. quasi-Baer) ⇒ σ-Baer (resp. σ-quasi-Baer); right (or left) p.q.-Baer (resp. right (or left) p.p.) ⇒ right (or left) σ-p.q.-Baer (resp. right (or left) σ-p.p.); σ-Baer ⇒ σ-quasi-Baer ⇒ σ-p.q.-Baer. All the implications are strict by the following examples;

Example 1. [9, Example 9] Let $Z$ be the ring of integers and consider the ring $Z \oplus Z$ with the usual addition and multiplication. Then the subring $R = \{(a, b) \in Z \oplus Z \mid a \equiv b \pmod{2}\}$ of $Z \oplus Z$ is a commutative reduced ring which has only two idempotents $(0, 0)$ and $(1, 1)$. Observe
that $R$ is not p.p. (and then $R$ is not Baer). Indeed, for $a = (2, 0) \in R$, $r_R(a) = (0) \oplus 2\mathbb{Z}$ which is not generated by an idempotent of $R$. Since $R$ is reduced, $R$ is not p.q.-Baer and hence it is not quasi-Baer. Let $\sigma : R \to R$ be a map defined by $\sigma((a, b)) = (b, a)$ for all $(a, b) \in R$. Then $\sigma$ is an endomorphism of $R$. Note that all the $\sigma$-sets of $R$ are $S \oplus S$ for some subset $S$ of $\mathbb{Z}$. Let $T = S \oplus S$. If $T = (0)$, then $r_R(T) = R = (1, 1)R$. If $T \neq (0)$, then $r_R(T) = (0) = (0, 0)R$. Hence $R$ is $\sigma$-Baer, and so $R$ is $\sigma$-quasi-Baer, $\sigma$-p.q.-Baer and $\sigma$-p.p.

Example 2. Let $\mathbb{Z}$ be the ring of integers. Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$ be the upper $2 \times 2$ triangular matrix ring over $\mathbb{Z}$. Since $\mathbb{Z}$ is quasi-Baer, $R$ is quasi-Baer by [12, Proposition 9]. But it is neither left p.p. nor right p.p. by [7, Example 8.1] and hence it is not p.p.. Consider an endomorphism $\sigma : R \to R$ given by

$$
\sigma \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}
$$

for all $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in R$.

We claim that $R$ is $\sigma$-p.p. but it is not $\sigma$-Baer. First, note that every $\sigma$-element of $R$ is of the form

$$
\alpha = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}.
$$

Let $\beta = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in r_R(\alpha)$ be arbitrary. Then $\alpha \beta = \begin{pmatrix} ax & ay \\ 0 & cz \end{pmatrix} = 0$.

Consider the following four cases;

(i) If $a$ and $c \neq 0$, then $x = y = z = 0$. Thus $r_R(\alpha) = (0)$, which is generated by idempotent 0 of $R$.

(ii) If $a \neq 0$ and $c = 0$, then $x = y = 0$ and $z$ is arbitrary. Thus

$$
r_R(\alpha) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & z \end{pmatrix} \in R \right\} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} R,
$$

i.e., it is generated by an idempotent $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ of $R$.

(iii) If $a = 0$ and $c \neq 0$, then $x, y$ are arbitrary and $z = 0$. Thus

$$
r_R(\alpha) = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \in R \right\} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} R,
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$$
r_R(\alpha) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & z \end{pmatrix} \in R \right\} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} R,
$$

i.e., it is generated by an idempotent $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ of $R$.

(iii) If $a = 0$ and $c \neq 0$, then $x, y$ are arbitrary and $z = 0$. Thus

$$
r_R(\alpha) = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \in R \right\} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} R,
$$
i.e., it is generated by an idempotent \( \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \) of \( R \).

(iv) If \( a \) and \( c = 0 \), then \( x, y \) and \( z \) are arbitrary. Thus \( r_R(\alpha) = R \), which is generated by idempotent 1 of \( R \). Hence \( R \) is a right \( \sigma \)-p.p. ring. Similarly, we can show that \( R \) is a left \( \sigma \)-p.p. ring.

Consequently, \( R \) is a \( \sigma \)-p.p. ring.

Example 3. [6, Example 1.3] Let \( Z_2 \) be the field of two elements and consider \( R = \{ (x_n) \in \prod_{n=1}^{\infty} Z_2 \mid x_n \text{ is eventually constant} \} \). Then \( R \) is a Boolean ring which is not self-injective. By \([12, p.79, p.249 \text{ and p.250}], R \text{ is not Baer and hence it is not quasi-Baer since } R \text{ is reduced. But } R \text{ is p.q.-Baer and hence it is p.p. since } R \text{ is reduced.}

(1) Let \( \sigma_1 : R \to R \) be defined by \( \sigma_1((x_1, x_2, \ldots)) = (x_2, x_3, \ldots) \). Then \( \sigma_1 \) is an endomorphism of \( R \). Note that the \( \sigma_1 \)-ideals of \( R \) are only \( R \) and \( (0) \). Hence \( R \) is \( \sigma_1 \)-quasi-Baer.

(2) Let \( \sigma_2 : R \to R \) be defined by \( \sigma_2((x_1, x_2, x_3, \ldots)) = (0, x_2, x_3, \ldots) \). Then \( \sigma_2 \) is an endomorphism of \( R \). Note that every ideal of \( R \) is a \( \sigma_2 \)-ideal of \( R \). Hence \( R \) is not \( \sigma_2 \)-quasi-Baer. But \( R \) is \( \sigma_2 \)-p.q.-Baer.

(3) Let \( \sigma_3 : R \to R \) be defined by \( \sigma_3((x_1, x_2, x_3, \ldots)) = (x_2, x_1, x_3, \ldots) \) and consider a projection \( \pi : R \to R \) given by \( \pi((x_1, x_2, \ldots)) = (x_3, x_4, \ldots) \). Then \( \sigma_3 \) is an endomorphism of \( R \). Note that every ideal of \( R \) is not always \( \sigma_3 \)-ideal of \( R \), for example, \((0) \times Z_2 \times \pi(I) \) is an ideal of \( R \) for some ideal \( I \) of \( R \) but it is not \( \sigma_3 \)-ideal of \( R \). On the other hand, for any ideal \( I \) of \( R \), \( J = Z_2 \times Z_2 \times \pi(I) \) and \( K = (0) \times (0) \times \pi(I) \) are \( \sigma_3 \)-ideals of \( R \). Then \( r_R(J) = (0) \times (0) \times r_R(\pi(I)) \) and \( r_R(K) = Z_2 \times Z_2 \times r_R(\pi(I)) \). Since \( R \) is not quasi-Baer, \( \pi(R) \) is not quasi-Baer and so \( R \) is not \( \sigma_3 \)-quasi-Baer. But \( R \) is \( \sigma_3 \)-p.q.-Baer.

We begin with the following lemmas;

Lemma 1.1. Let \( R \) be a ring with an endomorphism \( \sigma \). Then

(1) If \( I \) is a right \( \sigma \)-ideal of \( R \), then \( RI \) is a right \( \sigma \)-ideal of \( R \);

(2) If \( I \) is a left \( \sigma \)-ideal of \( R \), then \( IR \) is a left \( \sigma \)-ideal of \( R \).

Proof. (1) Let \( I \) be a right \( \sigma \)-ideal of \( R \). Clearly, \( RI \) is a right ideal of \( R \). Let \( t \in RI \) be arbitrary. Then \( t = \sum_{i=1}^{n} a_i b_i \) for some \( a_i \in R \), \( b_i \in I \) and some integer \( n \in Z^+ \). Since \( I \) is a right \( \sigma \)-ideal of \( R \), \( \sigma(I) \subseteq I \). For each \( i \), \( \sigma(a_i b_i) = \sigma(a_i) \sigma(b_i) \in RI \), and so \( \sigma(RI) \subseteq RI \). Hence \( RI \) is a right \( \sigma \)-ideal of \( R \).

(2) It follows from the similar argument given as in (1). □
Lemma 1.2. Let $R$ be a ring with an endomorphism $\sigma$. Then $R$ is $\sigma$-quasi-Baer if and only if the right annihilator of every right $\sigma$-ideal of $R$ is generated by an idempotent.

Proof. For any right $\sigma$-ideal $I$ of $R$, $RI$ is a $\sigma$-ideal of $R$ and $r_R(I) = r_R(RI)$ since $R$ has an identity. □

Lemma 1.3. Let $R$ be a $\sigma$-rigid ring. Then $R$ is $\sigma$-Baer if and only if $R$ is $\sigma$-quasi-Baer.

Proof. (⇒) Clear.

(⇐) Suppose that $R$ is $\sigma$-quasi-Baer. Let $S$ be any $\sigma$-set of $R$. Consider the right ideal $<S>$ of $R$ generated by $S$. Since $S$ is a $\sigma$-set of $R$, $<S>$ is a right $\sigma$-ideal of $R$. Since $R$ is $\sigma$-quasi-Baer, $r_R(<S>) = eR$ for some idempotent $e \in R$ by Lemma 1.2. We will show that $r_R(S) = r_R(<S>)$. Clearly, $r_R(<S>) \subseteq r_R(S)$. Let $b = \sum_{i=1}^{n} s_i x_i \in <S>$ be arbitrary. If $a \in r_R(S)$, then $s_i a = 0$ for all $s_i \in S$. Since $R$ is reduced, $s_i a = 0$ if and only if $as_i = 0$ if and only if $s_i Ra = 0$. Then $0 = \sum_{i=1}^{n} (as_i)x_i = \sum_{i=1}^{n} (s_i x_i)a = ba$, and so $a \in r_R(<S>)$. Thus $r_R(S) = r_R(<S>) = eR$. Hence $R$ is $\sigma$-Baer. □

Corollary 1.4. Let $R$ be a reduced ring. Then $R$ is Baer if and only if $R$ is quasi-Baer.

Proof. It follows from Lemma 1.3 by letting $\sigma = 1$. □

Lemma 1.5. Let $R$ be a $\sigma$-rigid ring. Then the following statements are equivalent:

1. $R$ is a right $\sigma$-p.p.-ring;
2. $R$ is a $\sigma$-p.p.-ring;
3. $R$ is a right $\sigma$-p.q.-Baer ring;
4. $R$ is a $\sigma$-p.q.-Baer ring;
5. For any $\sigma$-element $a \in R$ and any positive integer $n$, $r_R(a^n R) = eR$ for some idempotent $e \in R$.

Proof. Since $R$ is $\sigma$-rigid, $r_R(a) = l_R(a) = r_R(aR) = l_R(Ra) = r_R(a^n R)$ for any $\sigma$-element $a \in R$ and any positive integer $n$. Hence we have the result. □

In [1], Armendariz has shown that if $R$ is reduced, then $R$ is a Baer ring if and only if the polynomial ring $R[x]$ is a Baer ring. In this paper, we will generalize the result by showing that if $R$ is $\sigma$-rigid, then $R$ is
a $\sigma$-quasi-Baer ring if and only if the skew polynomial ring $R[x; \sigma]$ is a quasi-Baer ring; $R$ is a right (or left) $\sigma$-p.q.-Baer ring if and only if the skew polynomial ring $R[x; \sigma]$ is a right (or left) p.q.-Baer ring.

**Lemma 1.6.** Let $R$ be a $\sigma$-rigid ring. Then for all $a, b, c,$ and $d \in R$,

1. $a \sigma(b) = 0$ if and only if $\sigma(b)a = 0$;
2. If $ab = 0$ and $bc + da = 0$, then $bc = da = 0$;
3. If $ab = 0$ and $ad + cb = 0$, then $ad = cb = 0$;
4. If $ab = 0$, then $a \sigma(b) = \sigma(a)b = 0$;
5. If $a \sigma^k(b) = 0$ for some positive integer $k$, then $ab = 0$.

**Proof.** (1) is clear.

(2) If $ab = 0$ and $bc + da = 0$, then $0 = (bc + da)b = (bc)b + (da)b = bc$, and so $bc = 0$. Hence $da = 0$.

(3) It is similar to the proof of (2).

(4) Suppose that $ab = 0$. Since $R$ is reduced, $ba = 0$. Thus

$$a \sigma(b) \sigma(a \sigma(b)) = a \sigma(ba) \sigma^2(b) = 0.$$ 

Since $R$ is $\sigma$-rigid, $a \sigma(b) = 0$. Similarly, if $ab = 0$, then $\sigma(a)b = 0$.

(5) If $a \sigma^k(b) = 0$ for some positive integer $k$, then by using (4) repeatedly we have $\sigma^k(ab) = \sigma^k(\sigma(a))\sigma^k(b) = 0$, and so $ab = 0$ because $\sigma$ is a monomorphism. $\square$

For a ring $R$ with an endomorphism $\sigma$, there exists an endomorphism of $A = R[x; \sigma]$ which extends $\sigma$. For example, consider a map $\tilde{\sigma}$ on $A$ defined by $\tilde{\sigma}(f(x)) = \sigma(a_0) + \sigma(a_1)x + \cdots + \sigma(a_n)x^n$ for all $f(x) = a_0 + a_1x + \cdots + a_nx^n \in A$. Then $\tilde{\sigma}$ is an endomorphism of $A$ and $\tilde{\sigma}(a) = \sigma(a)$ for all $a \in R$, which means that $\tilde{\sigma}$ is an extension of $\sigma$. We call the endomorphism of $A = R[x; \sigma]$ which extends $\sigma$ an extended endomorphism of $\sigma$. Let $\Sigma_\sigma$ be the set of all extended endomorphisms on $A$ of $\sigma$. Note that $\Sigma_\sigma \neq \emptyset$ since $\tilde{\sigma} \in \Sigma_\sigma$.

**Lemma 1.7.** Let $R$ be a ring with an endomorphism $\sigma$ and let $\Sigma_\sigma$ be the set of all extended endomorphisms on $A = R[x; \sigma]$ of $\sigma$. Then

1. If $I$ is a $\sigma$-ideal of $R$, then $IA$ is a $\theta$-ideal of $A$ for all $\theta \in \Sigma_\sigma$;
2. If $I$ is a right principal $\sigma$-ideal of $R$, then $IA$ is a right principal $\theta$-ideal of $A$ for all $\theta \in \Sigma_\sigma$;
3. If $I$ is a left principal $\sigma$-ideal of $R$, then $AI$ is a left principal $\theta$-ideal of $A$ for all $\theta \in \Sigma_\sigma$.  

Proof. It is straightforward. □

Lemma 1.8. Let $R$ be a ring with an endomorphism $\sigma$ and let $\Sigma_\sigma$ be the set of all extended endomorphisms on $A = R[x; \sigma]$ of $\sigma$. Then $R$ is $\sigma$-rigid if and only if $A$ is $\theta$-rigid for all $\theta \in \Sigma_\sigma$. In this case, $\sigma(e) = e$ for every idempotent $e \in R$.

Proof. Assume that $R$ is $\sigma$-rigid and $A$ is not $\theta$-rigid for some $\theta \in \Sigma_\sigma$. Then there exists a nonzero $f \in A$ such that $f\theta(f) = 0$. Since $R$ is $\sigma$-rigid, $f \not\in R$. Let $f = \sum_{i=0}^{m} a_i x^i$ where $a_i \in R, a_m \neq 0$ for some $m \geq 1$. Since $f\theta(f) = 0$, $a_m \sigma^m(a_m) = 0$. Since $R$ is $\sigma$-rigid, $a_m^2 = 0$ by Lemma 1.6, and then $a_m = 0$ since $R$ is reduced, a contradiction. Hence $A$ is $\theta$-rigid for all $\theta \in \Sigma_\sigma$. The converse is true by the definition of extended endomorphism of $\sigma$. Let $e$ be any idempotent of $R$. In case that $A$ is $\theta$-rigid for each $\theta \in \Sigma_\sigma$ (and then $A$ is reduced). Hence $e$ is central idempotent in $A$, and thus $ce = xe = \sigma(e)x$, which implies that $\sigma(e) = e$. □

Note that for a reduced ring $R$, $A = R[x; \sigma]$ is not necessarily reduced. Indeed, consider the reduced ring $R$ and $\sigma$ introduced in Example 1. Let $f = (0, 2)x \in A$. Then $f^2 = (0, 2)x(0, 2)x = (0, 2)(0, 2)x^2 = (0, 2)(2, 0)x^2 = (0, 0)x^2 = 0$. But $f \neq 0$. Hence $A$ is not reduced.

We need the following corollary as a special case of [9, Proposition 6].

Corollary 1.9. Let $R$ be a $\sigma$-rigid ring. Then for any

$$
f = \sum_{i=0}^{m} a_i x^i, g = \sum_{j=0}^{n} b_j x^j \in R[x; \sigma],
$$

$fg = 0$ if and only if $a_i b_j = 0$ for each $i, j$. 

2. Skew polynomial rings over $\sigma$-quasi-Baer and $\sigma$-p.q.-Baer rings

We recall from [2] an idempotent $e \in R$ is left (resp. right) semicentral in $R$ if $eae = ae$ (resp. $eae = ea$), for all $a \in R$. Equivalently, an idempotent $e \in R$ is left (resp. right) semicentral if $eR$ (resp. $Re$) is an ideal of $R$. Since the right annihilator of a right $\sigma$-ideal is an ideal, we can note that the right annihilator of a right $\sigma$-ideal is generated by a left semicentral idempotent in a $\sigma$-quasi-Baer ring. Observe that
if \(e_1, e_2, \ldots, e_m\) are left (or right) semicentral idempotents of \(R\), then \(e = e_1e_2 \cdots e_m\) is an idempotent of \(R\). Thus we can obtain the following lemma:

**Lemma 2.1.** Let \(R\) be a ring with an endomorphism \(\sigma\). Then \(R\) is a right (resp. left) \(\sigma\)-p.q.-Baer if and only if the right (resp. left) annihilator of every finitely generated right (resp. left) \(\sigma\)-ideal of \(R\) is generated by an idempotent of \(R\).

**Proof.** It is enough to show the left-handed version because the right-handed version is similarly proved. Suppose that \(R\) is right \(\sigma\)-p.q.-Baer and let \(I = \sum_{i=1}^m a_i R\) be any finitely generated right \(\sigma\)-ideal of \(R\). Then \(r_R(I) = \bigcap_i e_i R\) where \(r_R(a_i R) = e_i R\). By the above observation, \(r_R(I)\) is an ideal of \(R\) and \(e_i\) is a left semicentral idempotent of \(R\). Since each \(e_i\) is left semicentral idempotents of \(R\), \(e = e_1e_2 \cdots e_m\) is idempotent of \(R\), and so \(r_R(I) = e R\). The converse is clear. \(\square\)

**Lemma 2.2.** Let \(R\) be a \(\sigma\)-rigid ring. If \(e \in R\) is a left semicentral idempotent, then \(e\) is also a left semicentral idempotent in \(R[x; \sigma]\).

**Proof.** Let \(f = \sum_{i=0}^m a_i x^i \in R[x; \sigma]\) be arbitrary. Since \(R\) is \(\sigma\)-rigid, \(\sigma(e) = e\) for any idempotent \(e \in R\) by Lemma 1.8. Since \(e\) is a left semicentral idempotent, \(ea_i e = a_i e\) for each \(i\). Then \(fe = \sum_{i=0}^m a_i \sigma^i(e)x^i = \sum_{i=0}^m a_i e x^i = \sum_{i=0}^m ea_i e x^i = e f e\). Hence \(e\) is a left semicentral idempotent in \(R[x; \sigma]\). \(\square\)

**Theorem 2.3.** Let \(R\) be a ring with an endomorphism \(\sigma\) and let \(\Sigma_\sigma\) be the set of all extended endomorphisms on \(A = R[x; \sigma]\) of \(\sigma\). If \(R\) is \(\sigma\)-rigid, then the following are equivalent:

1. \(R\) is \(\sigma\)-quasi-Baer;
2. \(A\) is quasi-Baer;
3. \(A\) is \(\theta\)-quasi-Baer for all \(\theta \in \Sigma_\sigma\).

**Proof.** (1) \(\Rightarrow\) (2). Suppose that \(R\) is \(\sigma\)-quasi-Baer. Let \(I\) be an arbitrary ideal of \(A\). If \(g \in r_A(I)\), then \(fg = 0\) for all \(f \in I\). Let \(f = \sum_{i=0}^m a_i x^i, g = \sum_{j=0}^n b_j x^j\). Then by Corollary 1.9, \(a_i b_j = 0\) for all \(i, j\). Consider the set \(I_c\) of all coefficients of polynomials in \(I\). Then \(I_c\) is an ideal of \(R\) and \(b_0, b_1, \ldots, b_n \in r_R(I_c)\). We can observe that \(I_c\) is an \(\sigma\)-ideal of \(R\). Indeed, for any \(f = \sum_{i=0}^m a_i x^i \in I, xf = \sum_{i=0}^{m+1} \sigma(a_i) x^i\), and \(\sigma(a_i) \in I_c\) for each \(i\). Thus \(I_c\) is a \(\sigma\)-ideal of \(R\). Since \(R\) is \(\sigma\)-quasi-Baer and \(I_c\) is a \(\sigma\)-ideal of \(R, r_R(I_c) = e R\) for some idempotent \(e \in R\). Thus \(g = ge\) and hence \(r_A(I) \subseteq e A\). Now \(I_c e = 0\). Since \(\sigma(e)=\)
e, by Lemma 1.8, we have $\text{I}e = 0$ so $eA \subseteq r_A(I)$. Therefore $r_A(I) = eA$. Hence $A$ is quasi-Baer.

(2) $\Rightarrow$ (3). It is clear.

(3) $\Rightarrow$ (1). Suppose that $A$ is $\theta$-quasi-Baer for all $\theta \in \Sigma_\sigma$. Let $I$ be any $\sigma$-ideal of $R$. Then by Lemma 1.7, $IA$ is a $\theta$-ideal of $A$. Since $A$ is $\theta$-quasi-Baer, $r_A(IA) = eA$ for some semicentral idempotent $e \in A$. Since $A$ is $\theta$-rigid (and so $A$ is reduced) by Lemma 1.8, $e$ is a central idempotent in $A$, and hence $e$ is an idempotent in $R$ by [10, Theorem 3.15]. Since $r_R(I) = r_A(IA) \cap R = eR$, $R$ is $\sigma$-quasi-Baer. □

Remark. (1) If $\sigma$ is an automorphism, we can check the condition “$R$ is $\sigma$-rigid” does not need by using a similar method in the proof of Theorem 1.2 in [6]. (2) there is an example of a $\sigma$-quasi-Baer ring $R$ and an endomorphism $\sigma$ of $R$ such that $R[x; \sigma]$ is not quasi-Baer (refer Example 1.4 in [6]).

Corollary 2.4. Let $R$ be a ring with an endomorphism $\sigma$ and let $\Sigma_\sigma$ be the set of all extended endomorphisms on $A = R[x; \sigma]$ of $\sigma$. If $R$ is $\sigma$-rigid, then the following are equivalent:

(1) $R$ is $\sigma$-Baer;
(2) $A$ is Baer;
(3) $A$ is $\theta$-quasi-Baer for all $\theta \in \Sigma_\sigma$.

Proof. It follows from Lemma 1.3 and Theorem 2.3. □

Corollary 2.5. [1, Theorem A] Let $R$ be a reduced ring and let $A = R[x]$. Then $R$ is Baer if and only if $R[x]$ is Baer.

Proof. It follows from Corollary 1.4 and Corollary 2.4. □

Theorem 2.6. Let $R$ be a ring with an endomorphism $\sigma$ and let $\Sigma_\sigma$ be the set of all extended endomorphisms on $A = R[x; \sigma]$ of $\sigma$. If $R$ is $\sigma$-rigid, then the following are equivalent:

(1) $R$ is right $\sigma$-p.q.-Baer;
(2) $R$ is $\sigma$-p.q.-Baer;
(3) $A$ is right $p.q.$-Baer;
(4) $A$ is $p.q.$-Baer;
(5) $A$ is $\theta$-p.q.-Baer for all $\theta \in \Sigma_\sigma$;
(6) $A$ is right $\theta$-p.q.-Baer for all $\theta \in \Sigma_\sigma$. 

Proof. (1) $\iff$ (2) follows from Lemma 1.5. (3) $\iff$ (4) also follows from Lemma 1.5 by letting $\sigma = 1$. (4) $\Rightarrow$ (5) $\Rightarrow$ (6) is clear. It remains to show that (1) $\Rightarrow$ (3) and (6) $\Rightarrow$ (1).

(1) $\Rightarrow$ (3). Suppose that $R$ is right $\sigma$-p.p.-Baer. Let $I$ be any right principal ideal of $A$ generated by $h = \sum_{k=0}^{n} a_k x^k$. If $g \in r_A(I)$, then $fg = 0$ for all $f \in I$. Let $f = \sum_{i=0}^{l} c_i x^i, g = \sum_{j=0}^{m} b_j x^j$. Then by Lemma 1.6, $c_i b_j = 0$ for all $i, j$. Let $I_c$ be the set of all coefficients of all $f \in I$. Note that $I_c$ is a right $\sigma$-ideal of $R$ and $b_0, b_1, \ldots, b_n \in r_R(I_c)$ as given in the proof of Theorem 2.3. Since $I$ is a right principal ideal of $A$, $I_c$ is a right finitely generated ideal of $R$ with a generating set \{0, \ldots, n\}. Since $R$ is right $\sigma$-p.q.-Baer and $I_c$ is a right finitely generated $\sigma$-ideal of $R$, $r_R(I_c) = eR$ for some idempotent $e$ of $R$ by Lemma 2.1. Hence $r_A(I) = eA$, and so $A$ is right p.q.-Baer.

(6) $\Rightarrow$ (1). Suppose that $A$ is right $\theta$-p.q.-Baer for all $\theta \in \Sigma_\sigma$. Let $I$ be any right principal $\sigma$-ideal of $R$. Then by Lemma 1.1, $IA$ is a right principal $\theta$-ideal of $A$. Since $A$ is $\theta$-p.q.-Baer, $r_A(IA) = eA$ for some semicentral idempotent $e \in A$. Since $A$ is $\theta$-rigid (and so reduced) by Lemma 1.8, $e$ is a central idempotent in $A$, and hence $e$ is an idempotent in $R$ by [10, Theorem 3.15]. Since $r_R(I) = r_A(IA) \cap R = eR$, $R$ is right $\sigma$-p.q.-Baer. □

Corollary 2.7. Let $R$ be a ring with an endomorphism $\sigma$ and let $\Sigma_\sigma$ be the set of all extended endomorphisms on $A = R[x; \sigma]$ of $\sigma$. If $R$ is $\sigma$-rigid, then the following are equivalent:

1. $R$ is right $\sigma$-p.p.;
2. $R$ is $\sigma$-p.p.;
3. $A$ is right p.p.;
4. $A$ is p.p.;
5. $A$ is $\theta$-p.p. for all $\theta \in \Sigma_\sigma$;
6. $A$ is right $\theta$-p.p. for all $\theta \in \Sigma_\sigma$.

Proof. It follows from the Lemma 1.5 and Theorem 2.6. □

Corollary 2.8. [1, Theorem B] Let $R$ be a reduced. Then $R$ is p.p.-Baer if and only if $R[x]$ is p.p.-Baer;

Proof. It follows from the Lemma 1.5 (by letting $\sigma = 1$) and Corollary 2.7. □

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References


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