LIMIT THEOREMS FOR PARTIAL SUM PROCESSES OF A GAUSSIAN SEQUENCE

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Abstract. In this paper we establish limsup and liminf theorems for the increments of partial sum processes of a dependent stationary Gaussian sequence.

1. Introduction and results

Let \( \{X_j; \ j = 1, 2, \ldots\} \) be a sequence of independent identically distributed (i.i.d) random variables and let \( S_0 = 0 \) and \( S_n = \sum_{j=1}^{n} X_j \). For an integer sequence \( \{a_n; \ n = 1, 2, \ldots\} \) with \( 1 \leq a_n \leq n \), put

\[
U_n = \max_{1 \leq k \leq n-a_n} (S_{k+a_n} - S_k).
\]

Csörgő and Révész [6] obtained the following strong limit law

\[
\lim_{n \to \infty} \frac{U_n}{b_n} = 1 \quad \text{a.s.}
\]

under some conditions of \( \{X_j\} \) and \( \{a_n\} \), where \( \{b_n; \ n = 1, 2, \ldots\} \) is some sequence of constants. For further various results on this limit law (1.1) about the sequence of i.i.d. random variables, we refer to ([5], [7], [8], [9], [11], [19], [20], [21]).

On the other hand, Lin ([15], [16], [18]) established large increment results for a sequence of independent or mixing dependent random variables. Theoretically and practically, strong dependent sequences are...
important and interesting. Usually one considers the case of Gaussian sequences.

Horváth and Shao [10] studied extreme value limit distributions for the maximum of partial sums of a stationary Gaussian sequence with long-range dependence.

Recently, Csáki and Gonchigdanzan [4] investigated almost sure central limit theorems for the maximum of dependent stationary Gaussian sequences.

In this paper we are interested in the strong limit law types as in (1.1) about partial sum processes of a dependent stationary Gaussian sequence. Let \( \{\xi_j; j = 1, 2, \ldots\} \) be a centered stationary Gaussian sequence with \( E\xi_1^2 = 1 \) and \( \rho_n = E(\xi_1\xi_{1+n}), \ n \geq 1 \). Put \( S_0 = 0, \ S_n = \sum_{j=1}^{n} \xi_j \) and \( \sigma(n) = \sqrt{ES_n^2} \). Assume that \( \sigma(n) \) can be extended to a continuous function \( \sigma(t) \) of \( t > 0 \) which is nondecreasing and regularly varying with exponent \( \alpha \) for some \( 0 < \alpha < 1 \). Suppose that \( \{a_n; n \geq 1\} \) is a sequence of positive integers such that

\( (i) \quad 1 \leq a_n \leq n \).

Denote \( \beta_n = (2(\log(n/a_n) + \log \log n))^{1/2} \) for \( n > e \).

Recently, Choi et al. [3] proved the following Theorems A and B.

**Theorem A.** Suppose that the sequence \( \{a_n; n \geq 1\} \) satisfies conditions (i) and

\( (ii) \quad \limsup_{n \to \infty} a_n/n =: \mu < 1, \)

\( (iii) \quad \text{there exist } 0 < \mu_2 \leq \mu_1 \leq 1 \text{ such that, for any } m < n, \text{ we have } \mu_1a_m \leq a_n \text{ and } \mu_2a_m/m \geq a_n/n. \)

Assume that, for \( n \geq 1 \), either

\( (iv) \quad \rho_n \leq 0 \)

or

\( (v) \quad |\rho_n| \leq \sigma^2(n)/n^2. \)

Then we have

\[
\limsup_{n \to \infty} \sup_{0 \leq i \leq n} \sup_{1 \leq j \leq a_n} \frac{|S_{i+j} - S_i|}{\sigma(a_n)\beta_n} = 1 \ a.s.,
\]

\[ 1.2 \]

\[
\limsup_{n \to \infty} \frac{|S_{n+a_n} - S_n|}{\sigma(a_n)\beta_n} = 1 \ a.s.
\]

Next, consider the case of a limit result.
Theorem B. Suppose that the condition (i) and one of (iv) and (v) are satisfied. Further suppose that

\[(vi) \lim_{n \to \infty} \frac{\log(n/a_n)}{\log \log n} = \infty.\]

Then we have

\[
\lim_{n \to \infty} \sup_{0 \leq i \leq n} \sup_{1 \leq j \leq a_n} \frac{|S_i + j - S_i|}{\sigma(a_n)\beta_n} = 1 \quad \text{a.s.,}
\]

\[
\lim_{n \to \infty} \sup_{0 \leq i \leq n} \frac{|S_i + a_n - S_i|}{\sigma(a_n)\beta_n} = 1 \quad \text{a.s.}
\]

Note that the condition (v) implies that, for \(n \geq 1,\)

\[-n^{2a-2}L(n) \leq \rho_n \leq n^{2a-2}L(n),\]

where \(L(n)\) is a slowly varying function.

For the Wiener process \(\{W(t), \ 0 \leq t < \infty\}\) with independent increments, Book and Shore [1] proved that \(\liminf\) results are different from \(\limsup\) results if the following condition

\[
\lim_{T \to \infty} \frac{\log(T/a_T)}{\log \log T} = \infty
\]

of Theorem 1.2.1 in [6] is replaced by

\[
\lim_{T \to \infty} \frac{\log(T/a_T)}{\log \log T} = r, \quad 0 \leq r < \infty.
\]

On this point of view, the main objective of this paper is to show that \(\liminf\) results are different from the results (1.2) and (1.3) for dependent Gaussian sequences if the condition (vi) is replaced by

\[
\lim_{n \to \infty} \frac{\log(n/a_n)}{\log_n n} = r, \quad 0 \leq r < \infty,
\]

where \(\theta = 1 + \varepsilon\) for \(\varepsilon > 0\) small enough.

The main results are as follows:
Theorem 1.1. If the condition (i) and
\[ \lim_{n \to \infty} \frac{\log(n/a_n)}{\log \log n} = r, \quad 0 \leq r \leq \infty \]
are satisfied, then we have
\[ \liminf_{n \to \infty} \sup_{0 \leq i \leq n} \sup_{1 \leq j \leq a_n} \frac{|S_{i+j} - S_i|}{\sigma(a_n)\beta_n} \leq \left( \frac{r}{1+r} \right)^{1/2} \text{ a.s.} \tag{1.4} \]

The following theorem is straightforward from Theorem 1.1.

Theorem 1.2. If the condition (i) and
\[ \lim_{n \to \infty} \frac{\log(n/a_n)}{\log \theta \log n} = r, \quad 0 \leq r \leq \infty \]
are satisfied, then we have
\[ \liminf_{n \to \infty} \sup_{0 \leq i \leq n} \sup_{1 \leq j \leq a_n} \frac{|S_{i+j} - S_i|}{\sigma(a_n)\beta'_n} \leq \left( \frac{r}{1+r} \right)^{1/2} \text{ a.s.,} \tag{1.5} \]
where \( \beta'_n = \{2(\log(n/a_n) + \log \log n)\}^{1/2} \) for \( n > e \).

Theorem 1.3. Suppose that conditions (i), (vii)' and one of (iv) and (v) are satisfied. Then we have
\[ \liminf_{n \to \infty} \sup_{0 \leq i \leq n} \frac{|S_{i+a_n} - S_i|}{\sigma(a_n)\beta'_n} \geq \left( \frac{r}{1+r} \right)^{1/2} \text{ a.s.} \tag{1.6} \]

Combining Theorems 1.2 and 1.3, we obtain the following liminf result.

Corollary 1.1. Under the assumptions of Theorem 1.3, we have
\[ \liminf_{n \to \infty} \sup_{0 \leq i \leq n} \sup_{1 \leq j \leq a_n} \frac{|S_{i+j} - S_i|}{\sigma(a_n)\beta'_n} = \left( \frac{r}{1+r} \right)^{1/2} \text{ a.s.,} \tag{1.7} \]
\[ \liminf_{n \to \infty} \sup_{0 \leq i \leq n} \frac{|S_{i+a_n} - S_i|}{\sigma(a_n)\beta'_n} = \left( \frac{r}{1+r} \right)^{1/2} \text{ a.s.} \]

Note that if \( r = \infty \) in (vii)', then (1.3) follows from (1.6) and Theorem 1.1 in Choi et al. [3]; if \( 0 \leq r < \infty \) in (vii)', then (1.7) differs from (1.2) under conditions (ii), (iii) and (vii)'.
2. Proofs of main theorems

The following Lemmas 2.1 and 2.2 are used for the proof of Theorem 1.1, and Lemma 2.1 is an analogue of Lemma 2.2 in [3] (See also Lemma 2.2 in [17]).

**Lemma 2.1.** For any $\varepsilon > 0$, there exists a positive constant $c_\varepsilon$ such that

$$\Pr\left\{ \sup_{0 \leq i \leq n} \sup_{1 \leq j \leq a_n} \frac{|S_{i+j} - S_i|}{\sigma(a_n)} \geq u \right\} \leq c_\varepsilon \frac{n}{a_n} \exp\left( -\frac{u^2}{2 + \varepsilon} \right)$$

for all $u > 1$.

The next Lemma 2.2 is obvious.

**Lemma 2.2.** Let $\{\xi, \xi_n; n \geq 1\}$ be a sequence of random variables. If

$$\Pr\{\xi_n > \xi\} \to 0 \quad \text{as} \quad n \to \infty,$$

then there is a subsequence $\{\xi_{n_k}\}$ such that

$$\limsup_{k \to \infty} \xi_{n_k} \leq \xi \quad \text{a.s.}$$

So we have

$$\liminf_{n \to \infty} \xi_n \leq \xi \quad \text{a.s.}$$

**Proof of Theorem 1.1.** First, suppose that $0 < r \leq \infty$. From (vii), there exists $\gamma > 0$ such that $n/a_n \geq (\log n)^\gamma$ for sufficiently large $n$. Thus by Lemma 2.1 we have, for any $\varepsilon > 0$,

$$\Pr\left\{ \sup_{0 \leq i \leq n} \sup_{1 \leq j \leq a_n} \frac{|S_{i+j} - S_i|}{\sigma(a_n)} \left\{ 2 \log \left( \frac{n}{a_n} \right) \right\}^{1/2} > \sqrt{1 + \varepsilon} \right\}$$

$$\leq c_\varepsilon \frac{n}{a_n} \exp\left( -\frac{2 + 2\varepsilon}{2 + \varepsilon} \log \frac{n}{a_n} \right)$$

$$\leq c_\varepsilon (\log n)^{-\gamma\varepsilon/(2+\varepsilon)} \to 0 \quad \text{as} \quad n \to \infty.$$

It follows from Lemma 2.2 that

$$\liminf_{n \to \infty} \sup_{0 \leq i \leq n} \sup_{1 \leq j \leq a_n} \frac{|S_{i+j} - S_i|}{\sigma(a_n)} \left\{ 2 \log \left( \frac{n}{a_n} \right) \right\}^{1/2} \leq 1 \quad \text{a.s.}$$
Hence by (vii) we obtain

\[
\liminf_{n \to \infty} \sup_{0 \leq i \leq n} \sup_{1 \leq j \leq a_n} \frac{|S_{i+j} - S_i|}{\sigma(a_n) \beta_n} \\
= \liminf_{n \to \infty} \sup_{0 \leq i \leq n} \sup_{1 \leq j \leq a_n} \frac{|S_{i+j} - S_i|}{\sigma(a_n) \left( \frac{2 \log(n/a_n)}{2(\log(n/a_n) + \log \log n)} \right)^{1/2}} \\
\leq \sqrt{\frac{r}{1 + r}} \quad \text{a.s.}
\]

On the other hand, consider the case \( r = 0 \). It follows from (vii) that for any small \( \varepsilon > 0 \) we have

\[
\frac{n}{a_n} < (\log n)^{\varepsilon/(2+\varepsilon)}
\]

for \( n \) sufficiently large. Applying Lemma 2.1 again, we get

\[
P\left\{ \sup_{0 \leq i \leq n} \sup_{1 \leq j \leq a_n} |S_{i+j} - S_i| > \sqrt{\varepsilon} \right\} \\
\leq c_{\varepsilon} (\log n)^{-\varepsilon/(2+\varepsilon)} \to 0 \quad \text{as} \quad n \to \infty
\]

and hence Lemma 2.2 gives

\[
\liminf_{n \to \infty} \sup_{0 \leq i \leq n} \sup_{1 \leq j \leq a_n} \frac{|S_{i+j} - S_i|}{\sigma(a_n) \beta_n} \leq 0 \quad \text{a.s.}
\]

Combining (2.1) with (2.2) completes the proof of Theorem 1.1. \( \square \)

The following Lemmas 2.3-2.5 are essential to prove Theorem 1.3.

**Lemma 2.3.** (cf. Corollary 1.2.2 in [14]) Let \( \xi = (\xi_{ij}) \) and \( \eta = (\eta_{ij}) \), \( 1 \leq i \leq n, 1 \leq j \leq m \), be centered Gaussian random vectors such that

\[
E(\xi_{ij}) = E(\eta_{ij}) \quad \text{for all} \quad i, j,
\]

\[
E(\xi_{ij} \xi_{jk}) \leq E(\eta_{ij} \eta_{jk}) \quad \text{for all} \quad i, j, k,
\]

\[
E(\xi_{ij} \xi_{lk}) \geq E(\eta_{ij} \eta_{lk}) \quad \text{for all} \quad i \neq l, \text{ j and k}.\]

Then, for all real numbers \( \lambda_{ij} \),

\[
P\left\{ \bigcap_{i=1}^{n} \bigcup_{j=1}^{m} (\eta_{ij} > \lambda_{ij}) \right\} \leq P\left\{ \bigcap_{i=1}^{n} \bigcup_{j=1}^{m} (\xi_{ij} > \lambda_{ij}) \right\}.
\]
Lemma 2.4. ([12], [13]) Let $\xi_j (j = 1, 2, \ldots, n)$ be standardized normal random variables with $\text{Cov}(\xi_i, \xi_j) = \Lambda_{ij}$ such that $\delta = \max_{i \neq j} |\Lambda_{ij}| < 1$. Then for any real number $u$ and integers $1 \leq l_1 < l_2 < \cdots < l_k \leq n$ with $k \leq n$, we have

\begin{equation}
(2.3) \quad P\left\{ \max_{1 \leq j \leq k} \xi_{l_j} \leq u \right\} \leq (\Phi(u))^k + c \sum_{1 \leq i < j \leq k} |\rho_{ij}| \exp\left( -\frac{u^2}{1 + |\rho_{ij}|} \right),
\end{equation}

where $\rho_{ij} = \Lambda_{l_i l_j}$ and $c = c(\delta)$ is a constant independent of $n$ and $u$, and $\Phi(u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} \exp(-y^2/2) \, dy$.

Under the stationary condition on $\rho_{ij}$, we can estimate an upper bound for the second term of the right hand side of (2.3) as follows:

Lemma 2.5. ([2]) Let $\xi_j$, $\delta$, $k$ and $\rho_{ij}$ be as in Lemma 2.4. Assume that for some $\nu > 0$

$|\rho_{ij}| < |i - j|^{-\nu}$ for all $i \neq j$.

Put $u = \{(2 - \eta) \log k\}^{1/2}$, where $0 < \eta < (1 - \delta)\nu/(1 + \nu + \delta)$. Then we have

$$
\sum := \sum_{1 \leq i < j \leq k} |\rho_{ij}| \exp\left( -\frac{u^2}{1 + |\rho_{ij}|} \right) \leq CK^{-\delta_0},
$$

where $\delta_0 = \{\nu(1 - \delta) - \eta(1 + \delta + \nu)\}/\{(1 + \nu)(1 + \delta)\} > 0$ and $c$ is a constant independent of $n$ and $u$.

Proof of Theorem 1.3. (1.6) is obvious when $r = 0$. In what follows, we assume that $0 < r \leq \infty$. For $\theta > 1$, let

$$
A_{k,l} = \{ n : \theta^{k-1} \leq n \leq \theta^k, \theta^{l-1} \leq a_n \leq \theta^l \},
$$

where $k = 1, 2, \ldots$; $l = 1, 2, \ldots$. The condition (vii)' implies that, for sufficiently large $k$, there exists $\gamma > 0$ such that

$$
1 \leq l \leq k + 1 - \gamma \log((k - 1) \log \theta)/\log \theta =: K
$$

and there exists $M > 0$ such that

$$
\theta(k, l) := [\theta^{k-l-1}/M] > 1.
$$
Noting that
\[
\lim_{n \to \infty} \sqrt{2 \log(n/a_n)} = \begin{cases} 
  (r/(1+r))^{1/2} & \text{if } 0 < r < \infty, \\
  1 & \text{if } r = \infty
\end{cases}
\]
by (vii)', then (1.6) is proved if we show that
\[
(2.4) \quad \lim \inf_{n \to \infty} \sup_{0 \leq i \leq n} \frac{|S_{i+a_n} - S_i|}{\sigma(a_n) \sqrt{2 \log(n/a_n)}} \geq 1 \quad \text{a.s.}
\]
By the regular variation of \(\sigma(\cdot)\), we have
\[
\sigma(\theta^{l-1}) \geq (\theta - 1)^{-\alpha} \sigma(\theta^l - \theta^{l-1})
\]
for some \(0 < \alpha < 1\). Thus
\[
\lim \inf_{n \to \infty} \sup_{0 \leq i \leq n} \frac{|S_{i+a_n} - S_i|}{\sigma(a_n) \sqrt{2 \log(n/a_n)}} \geq \lim \inf_{k \to \infty} \inf_{1 \leq l \leq K} \sup_{0 \leq i \leq \theta^l - 1} \frac{|S_{i+\theta^l} - S_i|}{\sigma(\theta^l) \sqrt{2 \log(\theta^k)} \leq R} \quad \text{a.s.}
\]
First, we will show that, for any small \(\varepsilon > 0\),
\[
(2.6) \quad J_2 \leq \varepsilon \quad \text{a.s.}
\]
We claim that, for some \(R > 2\),
\[
(2.7) \quad \lim \sup_{k \to \infty} \sup_{1 \leq i \leq K} \sup_{0 \leq i \leq \theta^k \theta^{l-1} \leq \theta^l} \frac{|S_{i+j} - S_{i+\theta^l}|}{\sigma(\theta^l - \theta^{l-1}) \sqrt{2 \log(\theta^k)} \leq R} \quad \text{a.s.}
\]
By the same way as the proof of Lemma 2.1, we can obtain
\[
P\left\{ \sup_{0 \leq i \leq \theta^k \theta^{l-1} \leq \theta^l} \frac{|S_{i+j} - S_{i+\theta^l}|}{\sigma(\theta^l - \theta^{l-1}) > u} \right\} \leq c_\varepsilon \theta^{k-l} e^{-u^2/(2+\varepsilon)}
\]
for all $u > 1$. Thus,

$$P\left\{ \sup_{1 \leq l \leq K} \sup_{0 \leq i \leq \theta^l} \sup_{0 \leq j \leq \theta^i} \frac{|S_{i+j} - S_{i+\theta^i}|}{\sigma(\theta^i - \theta^i - 1)} > R\sqrt{2 \log \theta^{k-l}} \right\}$$

$$\leq c \varepsilon \sum_{l=1}^{K} \theta^{-k-l} \exp \left( - \frac{8}{2 + \varepsilon} \log \theta^{k-l} \right)$$

$$\leq c \varepsilon \sum_{l=1}^{K} \theta^{-2(k-l)} \leq c \theta^{-\gamma/\log \theta}.$$  

Since $\gamma/ \log \theta > 1$, the Borel-Cantelli lemma implies (2.7), and thus (2.6) follows if $\theta \to 1$.

Next, consider $J_1$. For $0 \leq m \leq \theta(k, l)$, let

$$S(m) = S_{m \theta^l M + \theta^i} - S_{m \theta^l M}.$$  

It follows from (vii) that, for any $0 < \varepsilon < 1$,

$$P\left\{ \sup_{0 \leq i \leq \theta^l - 1} \frac{S_{i+\theta^i} - S_i}{\sigma(\theta^i) \sqrt{2 \log \theta^{k-l}}} \leq \sqrt{1 - \varepsilon} \right\}$$

$$\leq P\left\{ \max_{0 \leq m \leq \theta(k, l)} \frac{S(m)}{\sigma(\theta^i)} \leq \sqrt{1 - \varepsilon} \sqrt{2 \log \theta(k, l)} \right\}.$$

Assume that (iv) holds. By Lemma 2.3, we have

$$P\left\{ \max_{0 \leq m \leq \theta(k, l)} \frac{S(m)}{\sigma(\theta^i)} \leq \left( (2 - 2\varepsilon) \log \theta(k, l) \right)^{1/2} \right\}$$

$$\leq (\Phi\left( \left( (2 - 2\varepsilon) \log \theta(k, l) \right)^{1/2} \right))^{\theta(k,l)} \leq \exp \left( - c(\theta^{k-l})^\varepsilon \right),$$

where $c$ is a positive constant. Hence by (2.8) and (2.9), we have

$$P\left\{ \inf_{1 \leq l \leq K} \sup_{0 \leq i \leq \theta^l - 1} \frac{S_{i+\theta^i} - S_i}{\sigma(\theta^i) \sqrt{2 \log \theta^{k-l}}} \leq \sqrt{1 - \varepsilon} \right\}$$

$$\leq \sum_{l=1}^{K} \exp \left( - c(\theta^{k-l})^\varepsilon \right) \leq \exp \left( - c k^\varepsilon \gamma/\log \theta \right)$$

for all large $k$. It follows from the Borel-Cantelli lemma that

$$J_1 \geq 1 \quad \text{a.s.}$$
Consider the case when (v) holds. In this case, we can estimate an upper bound of the right hand side of (2.8) by Lemmas 2.4 and 2.5. Define

\[ r(m, m') = \text{Cov} \left( \frac{S(m)}{\sigma(\theta^l)}, \frac{S(m')}{\sigma(\theta^l)} \right), \quad m > m' = 0, 1, \ldots, \theta(k, l) \]

and let \( q = m - m' \geq 1 \). Then by (v) we have

\[
|r(m, m')| = \frac{1}{\sigma^2(\theta^l)} \left| E \left\{ \xi_{m\theta^l M} + \cdots + \xi_{m\theta^l M+1} + \cdots + \xi_{m' \theta^l M} + \cdots + \xi_{m' \theta^l M+1} \right\} \right| 
\]

\[
= \frac{1}{\sigma^2(\theta^l)} |\rho_{q^l M} + \cdots + \rho_{q^l M+1-\theta^l} + \cdots + \rho_{q^l M+\theta^l-1} + \cdots + \rho_{q^l M}| 
\]

\[
\leq \frac{(\theta^l)^2}{\sigma^2(\theta^l)} |\rho_{q^l M+1-\theta^l}| \leq \frac{\theta^{2l}}{\sigma^2(\theta^l)} \frac{\sigma^2(q^l M + 1 - \theta^l)}{(q^l M + 1 - \theta^l)^2} 
\]

\[
\leq \frac{1}{(qM - 1)^2} \frac{\sigma^2((qM - 1)\theta^l)}{\sigma^2(\theta^l)} \leq c(qM - 1)^{2\alpha-2} < q^{-\nu},
\]

where \( \nu = 1 - \alpha > 0 \). Applying Lemmas 2.4 and 2.5 for

\[
\xi_{l_j} = \frac{S(m)}{\sigma(\theta^l)}, \quad m = 0, 1, \ldots, \theta(k, l),
\]

\[
u = \left\{ (2 - \eta) \log \theta(k, l) \right\}^{1/2}, \quad \eta = 2\varepsilon,
\]

\[
|\rho_{l_j}| = |r(m, m')| < |m - m'|^{-\nu}, \quad m \neq m',
\]

then the right hand side of (2.8) is less than or equal to

\[
(\Phi(u))^{\theta(k, l)} + c(\theta(k, l))^{-\delta_0}.
\]

Thus we have

\[
P\left\{ \sup_{0 \leq t \leq \theta^k - 1} \frac{S_{t+\theta^l} - S_t}{\sigma(\theta^l) \sqrt{2 \log \theta^k - l}} \leq \sqrt{1 - \varepsilon} \right\} \leq \exp \left( -c(\theta^{k-l})\varepsilon \right) + c(\theta^{k-l})^{-\delta_0} \leq c \left( \theta^{k-l} \right)^{-\delta_0}
\]
for all large $k$. Considering $J_1$ in (2.5), we have

$$P \left\{ \inf_{1 \leq l \leq K} \sup_{0 \leq i \leq \theta_{l-1}} \frac{S_{l+\theta_i} - S_i}{\sigma(\theta_i) \sqrt{2 \log \theta_{k-l}}} \leq \sqrt{1 - \varepsilon} \right\} \leq K \sum_{l=1}^{K} c (\theta^{k-l})^{-\delta_0} \leq c k^{-\gamma \delta_0 / \log \theta}.$$ 

Thus the Borel-Cantelli lemma gives (2.10). From (2.5), (2.6) and (2.10), we obtain (2.4). This completes the proof of Theorem 1.3. □

ACKNOWLEDGEMENT. The authors wish to thank the referee for careful reading and helpful suggestion on this paper.

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