GIBBS PHENOMENON FOR TRIGONOMETRIC INTERPOLATION

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Abstract. The Gibbs’ phenomenon for the classical Fourier series is known. This occurs for almost all series expansions. This phenomenon has been observed even in sampling series. In this paper, we show the existence of Gibbs phenomenon for trigonometric interpolating polynomial by a simple and different manner from the work[4].

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1. Introduction

The trigonometric Fourier series and its convergence have been wee known. For a function with jump discontinuity, the graphs of the partial sums of the series exhibits ripples around the point of discontinuity. One may think the reason is that the number of items in partial sum is not big enough to provide a good approximation over the original infinite series. But we can observe the following fact: as we increase the number of items, ripples tend to approach a certain overshoot or downshoot near the discontinuity instead of disappearing. This phenomenon was first pointed out by [7] in 1898 and analyzed by [3] in 1899. For the history of this observation see [1]. This phenomenon is not the special quirk occurs only in trigonometric Fourier series. A similar phenomenon exists in other classical orthogonal series[5,10], spline expansions[2], wavelet series[6,8] and even in sampling approximations[9,11].

2. Gibbs phenomenon

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To make this paper a self-contained one, we represent the Gibbs’ phenomenon for Fourier series in detail. We begin with the definition of Gibbs phenomenon, which will be abbreviated as Gibbs throughout this paper.

**Definition 1.** Let $f$ be a with a jump discontinuity at $x_0$, i.e., the limits
$f(x_0^-) = \lim_{x \to x_0^+} f(x)$ and $f(x_0^+) = \lim_{x \to x_0^-} f(x)$ exist and are different.
Without loss of generality, we assume that $f(x_0^+) > f(x_0^-)$. We also let $(S_n f)(x)$ be a partial sum of the series expansion of $f$ for a given system.

Then we say that the series expansion of $f$ with respect to a given system shows a Gibbs at the right hand side of $x_0$ if there is a sequence $x_0 < x_n \to x_0$ such that $(S_n f)(x_n)$ converges to a number greater than $f(x_0^+)$ as $n \to \infty$.

Similarly, we speak of Gibbs at the left hand side of $x_0$ if there is a sequence $x_0 > x_n \to x_0$ such that $(S_n f)(x_n)$ converges to a number less than $f(x_0^-)$ as $n \to \infty$. We say a series expansion exhibits Gibbs at $x_0$ when it shows Gibbs at the right hand side of $x_0$ or at the left hand side of $x_0$.

**Remark.** (i) If an approximation system satisfies $(S_n f)(x - x_0) = (S_n g)(x)$ for $g$ defined by $g(x) = f(x - x_0)$, then we have the following fact; series expansion of $f$ shows Gibbs at $x_0$ if only if series expansion of $g$ shows Gibbs at 0.

(ii) In most cases, the translation property in (1) holds. One of exceptions is wavelet systems, where translation is eligible for dyadic rational numbers. So we try to show Gibbs at 0 as long as not mentioned otherwise.

**Gibbs in trigonometric Fourier series:**

Let $\varphi$ be the $2\pi$-periodic function defined by
$$
\varphi(x) = \begin{cases} 
-\frac{\pi}{2} - \frac{x}{2}, & -\pi < x < 0 \\
\frac{\pi}{2} - \frac{x}{2}, & 0 < x < \pi.
\end{cases}
$$

Then the Fourier series of $\varphi$ is
$$
\sum_{n=1}^{\infty} \frac{\sin n\tau x}{n} \tag{1}
$$
which agrees with $\varphi(x)$ for $x \neq 0, \pm\pi$, and is 0 at $x = 0, \pm\pi$. We notice $\varphi$ has jump discontinuities at $x = 0, \pm\pi$. We first wish to investigate the limit of partial sum as $x \to 0$. The series (1) can be written as
$$
\sum_{n=1}^{\infty} \frac{\sin n\tau x}{n} = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{\sin k\tau x}{k} = \lim_{n \to \infty} (S_n \varphi)(x).
$$
We further manipulate the partial sum as follows:

\[
(S_n \varphi)(x) = \sum_{k=1}^{n} \frac{\sin kx}{k} = \sum_{k=1}^{n} \int_{0}^{x} \cos kt \, dt
\]

\[
= \sum_{k=1}^{n} \int_{0}^{x} \left( \frac{1}{2} + \cos kt \right) \, dt - \frac{1}{2} x
\]

\[
= \int_{0}^{x} \sum_{k=1}^{n} \left( \frac{1}{2} + \cos kt \right) \, dt - \frac{1}{2} x
\]

\[
= \int_{0}^{x} D_n(t) \, dt - \frac{1}{2} x,
\]

where \( D_n \) is the "Dirichlet kernel"

\[
D_n(t) = \frac{\sin \left( n + \frac{1}{2} \right) t}{2 \sin \frac{t}{2}}.
\]

We also have

\[
\int_{0}^{x} D_n(t) \, dt = \int_{0}^{x} \left( \frac{\sin nt \cos \frac{1}{2} t}{2 \sin \frac{t}{2}} + \cos nt \right) \, dt
\]

\[
= \int_{0}^{x} \frac{\sin nt}{t} \, dt + \int_{0}^{x} \sin nt \left( \frac{\cos \frac{1}{2} t - \frac{1}{t}}{2 \sin \frac{t}{2}} \right) \, dt + \int_{0}^{x} \frac{\cos nt}{2} \, dt.
\]

Since

\[
\frac{\cos \frac{t}{2} - \frac{1}{t}}{2 \sin \frac{t}{2}} = \frac{1}{2} \cot \frac{t}{2} - \frac{1}{t}
\]

is a function of bounded variation over \((0, \pi)\) (see [10,(4.13)]), the second integral tends to 0 as \( n \to \infty \). By direct calculation we can see the third integral approaches 0 as \( n \to \infty \). Therefore, by a change of variable, we have

\[
(S_n \varphi)(x) = \int_{0}^{x} \frac{\sin t}{t} \, dt + o(1).
\]

For \( x > 0 \), replacing \( x \) by a sequence \( x_n = \frac{\pi}{n} \) and taking limit \( n \to \infty \) on both sides, we obtain

\[
\lim_{n \to \infty} (S_n \varphi)(x) = \int_{0}^{\pi} \frac{\sin t}{t} \, dt > \frac{\pi}{2},
\]

which implies there exists \textit{Gibbs} at the right had side of 0. By the symmetric argument we can show there also exists \textit{Gibbs} at the left hand side of 0. The ratio of overshoot to the jump is \( \frac{\pi}{2} \int_{0}^{\pi} \frac{\sin t}{t} \, dt = 1.17898 \cdots \), which is called \textit{Gibbs constant}.
We can simplify the whole procedure by taking a simpler step function. Let $(S_n F)(x)$ denote the $n$th partial sum of the Fourier series of 2-periodic function $F(x)$ defined by

$$F(x) = \begin{cases} 1, & 0 \leq x < 1 \\ -1, & -1 \leq x < 0 \end{cases}$$

Then $(S_n F)(x) \to 1$ as $n \to \infty$ for all $0 < x < 1$. However there is a positive sequence $x_n$ converges to 0 such that $(S_n F)(x_n)$ converges to a number greater than 1. Indeed, by taking $x_n = \frac{a}{n}$, we have

$$\lim_{n \to \infty} (S_n F)(x_n) = \frac{2}{\pi} \int_0^{\pi a} \frac{\sin t}{t} \, dt$$

and so that

$$\lim_{n \to \infty} (S_n F)\left(\frac{1}{n}\right) = \frac{2}{\pi} \int_0^{\pi} \frac{\sin t}{t} \, dt = 1.17898 \cdots > 1.$$  

We observe that Gibbs constants for these two functions $\varphi$ and $F$ are the same.

3. Trigonometric Fourier interpolating polynomial

We give a brief introduction of trigonometric interpolation polynomials. Let $x_k, k = 0, \ldots, 2n$, be points on the $x$-axis, distinct modulo $2\pi$. Then given any function $f(x)$ of period $2\pi$, the trigonometric interpolating polynomial which coincides with $f(x)$ at the points $x_k$ is equal to

$$\sum_{j=0}^{2n} f(x_j)t_j(x),$$  

where $t_j(x)$ is a trigonometric polynomial of order $n$ taking the value 1 when $x = x_j$ and the value 0 at the remaining points $x_k$.

The geometric structure of the points $x_k$ is of great importance here to get the sum converge to $f(x)$ as $n \to \infty$. Little is known about the geometric structure. For this reason, we shall be concerned mainly with the case of equidistant nodal points

$$x_j = x_0 + \frac{2\pi j}{2n+1}, \quad j = 0, 1, \ldots, 2n.$$  

The trigonometric polynomial coincide with the periodic function $f(x)$ at points (2) will be denoted by $(I_n f)(x)$ and will be called the $n$th trigonometric interpolating polynomial of $f$. 
The Dirichlet kernel

\[ D_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \cos kx = \frac{\sin \left( \frac{n + \frac{1}{2}}{2} \right)x}{2 \sin \frac{1}{2}x} \]

is a trigonometric polynomial of order \( n \) vanishing at the points \( \frac{2\pi j}{2n+1} \), \( j = 1, 2, \cdots, 2n \), and equal to \( n + \frac{1}{2} \) for \( x = 0 \).

Thus the polynomial \( D_n(x - x_j)/(n + \frac{1}{2}) \) is equal to 10 when \( x = x_j \) and to 0 at the remaining points \( x_k \). Therefore we have by (2)

\[ (I_nf)(x) = \frac{2}{2n + 2} \sum_{j=0}^{2n} f(x_j)D_n(x - x_j). \]

We have the following convergence property regarding trigonometric interpolating polynomial and it will be used in the following section.

**Proposition [12,(5.4)].** If \( f \) is of bounded variation, then \( (I_nf)(x) \) converges to \( f(x) \) at every point of continuity of \( f \). The convergence is uniform over every closed interval of continuity of \( f \).

**4. Gibbs phenomenon for trigonometric interpolating polynomials**

For Gibbs for trigonometric interpolation, there is a work of [4]. But our approach is much simpler and different because we start from different definition about Gibbs.

We show Gibbs appears for a \( 2\pi \) period function \( h \) defined by

\[ h(x) = \begin{cases} 1, & 0 \leq x < \pi \\ 0, & -\pi \leq x < 0, \end{cases} \]

which has a jump at 0.

**Lemma 1.** Trigonometric interpolating polynomial of \( h \) exhibits Gibbs at 0.

**Proof.** For this function the expression in (4) becomes

\[ (I_nh)(x) = \frac{1}{2(n + 1)} \sum_{j=0}^{n} \frac{\sin(n + 1/2)(x - x_j)}{\sin \frac{1}{2}(x - x_j)}. \]

Taking a sequence \( x_n = \frac{\pi}{2n+1} \) and replacing \( x \) by \( x_n \), we have
\[ (I_n h) \left( \frac{\pi}{2n+1} \right) = \frac{1}{2(n+1)} \sum_{j=0}^{n} \frac{\sin(n+1/2)}{\sin \frac{1}{2} \left( \frac{\pi}{2n+1} - \frac{2\pi j}{2n+1} \right)} \left( \frac{\pi}{2n+1} - \frac{2\pi j}{2n+1} \right). \]

This is an alternating series whose modulus is decreasing, and therefore is greater than the sum of the first two terms, i.e.,

\[ (I_n h) \left( \frac{\pi}{2n+1} \right) > \frac{1}{2n+1} \left\{ \frac{1}{\sin \frac{\pi}{2} \left( \frac{1}{2n+1} \right)} - \frac{1}{\sin \frac{\pi}{2} \left( \frac{-1}{2n+1} \right)} \right\}. \]

Since \( \sin x \leq x \) for \( 0 \leq x \leq \frac{\pi}{4} \), this sum is greater than

\[ \frac{1}{2n+1} \left\{ \frac{2}{\pi} \frac{1}{2n+1} \right\} = \frac{4}{\pi} > 1 = h(0^+), \quad \text{for all } n \geq 1. \]

Therefore trigonometric interpolating polynomial for \( h \) exhibits Gibbs at 0. We observe the overshoot of interpolation is greater than that of Fourier series. \( \square \)

**Remark.** If the jump is at one of the interpolating points, a possible ambiguity occurs in this sum depending on what we take the value of the function at this point to be. The only effect of this choice is to move the Gibbs from the right hand-side to the left or vise versa \([11]\).

Now we try to show Gibbs to exists for general functions other than our typical \( h \).

**Theorem 2.** Let \( f \) be a function of bounded variation and have a jump discontinuity at 0. Then the trigonometric interpolating polynomial of \( f \) exhibits Gibbs at 0.

**Proof.** Without loss of generality, we assume that the jump \( f(0^+) - f(0^-) := \alpha \) at 0 be positive, i.e., \( f(0^+) > f(0^-) \). Now we define \( g \) as

\[ g(x) = \begin{cases} f(x) - \alpha h(x), & x \neq 0 \\ f(0^-), & x = 0. \end{cases} \]

Then \( g(x) \) is continuous at 0 since \( \lim_{x \to 0^+} g(x) = f(0^-) = \lim_{x \to 0^-} g(x) \). By the proposition regarding convergence, the trigonometric interpolation of \( g \) converges uniformly on closed interval where \( g \) is continuous. By the linearity, the interpolation of \( g \) is given by
\[(I_n g)(x) = (I_n f)(x) - \alpha (I_n h)(x),\]

which converges to \(f(0^-)\) at \(x = 0\). Now we replace \(x\) by \(x_n = \frac{\pi}{2n+1}\) and letting \(n \to \infty\), we obtain

\[
\lim_{n \to \infty} \left\{ (I_n f)(x_n) - \lim_{n \to \infty} \alpha (I_n h)(x_n) \right\} = f(0^-),
\]

\[
\lim_{n \to \infty} (I_n f)(x_n) = f(0^-) + \alpha \lim_{n \to \infty} (I_n h)(x_n),
\]

\[
\lim_{n \to \infty} (I_n f)(x_n) = f(0^-) + \alpha L,
\]

where \(L\) is greater than 1. We find the quantity

\[
f(0^-) + \alpha L - f(0^+) = \alpha (L - 1)
\]

is positive. Consequently we have

\[
\lim_{n \to \infty} (I_n f)(x_n) > f(0^-)
\]

and this means that trigonometric interpolation of \(f\) exhibits Gibbs at 0. \(\square\)

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References

7. A. A. Michelson, Letter to the editor, Nature 58(1898) 544-545.
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