CONVERGENCE THEOREMS OF THE ITERATIVE SEQUENCES FOR NONEXPANSIVE MAPPINGS

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Abstract. In this paper, we will prove the following: Let $D$ be a nonempty subset of a normed linear space $X$ and $T : D \to X$ be a nonexpansive mapping. Let $\{x_n\}$ be a sequence in $D$ and $\{t_n\}$, $\{s_n\}$ be real sequences such that

(i) $0 \leq t_n \leq t < 1$ and $\sum_{n=1}^{\infty} t_n = \infty$,
(ii) (a) $0 \leq s_n \leq 1$, $s_n \to 0$ as $n \to \infty$ and $\sum_{n=1}^{\infty} t_n s_n < \infty$ or (b) $s_n = s$ for all $n \geq 1$ and $s \in [0, 1)$,
(iii) $x_{n+1} = (1-t_n)x_n + t_n T(s_n Tx_n + (1-s_n)x_n)$ for all $n \geq 1$.

Then, if the sequence $\{x_n\}$ is bounded, then

$$\lim_{n \to \infty} \|x_n - Tx_n\| = 0.$$ 

This result improves and complements a result of Deng [2]. Furthermore, we will show that certain conditions on $D$, $X$ and $T$ guarantee the weak and strong convergence of the Ishikawa iterative sequence to a fixed point of $T$.

Let $X$ be a real normed linear space and $D$ a nonempty subset of $X$. Let $T : D \to X$ be nonexpansive, i.e.,

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in D$.

In 1976, Ishikawa [4] proved the following interesting result:

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Theorem I. Let $D$ be a nonempty subset of a normed linear space $X$ and $T : D \rightarrow X$ be a nonexpansive mapping. Let $\{x_n\}$ and $\{t_n\}$ be a sequence in $D$ and a sequence of real numbers, respectively, such that

(i) $0 \leq t_n \leq t < 1$ and $\sum_{n=1}^{\infty} t_n = \infty$,
(ii) $x_{n+1} = (1-t_n)x_n + t_n Tx_n$ for all $n \geq 1$.

If $\{x_n\}$ is bounded, then

$$\lim_{n \to \infty} \|x_n - Tx_n\| = 0.$$ 

In 1996, Deng [2, Theorem 1] extended Theorem I to the Ishikawa iterative sequence by proving the following:

Theorem D. Let $D$ be a nonempty subset of a normed linear space $X$ and $T : D \rightarrow X$ be a nonexpansive mapping. Let $\{x_n\}$ and $\{t_n\}$, $\{s_n\}$ be a sequence in $D$ and sequences of real numbers, respectively, such that

(i) $0 \leq t_n \leq t < 1$ and $\sum_{n=1}^{\infty} t_n = \infty$,
(ii) $0 \leq s_n \leq 1$ and $\sum_{n=1}^{\infty} s_n < \infty$, 
(iii) $x_{n+1} = (1-t_n)x_n + t_n T(s_n Tx_n + (1-s_n)x_n)$ for all $n \geq 1$.

If $\{x_n\}$ is bounded, then

$$\lim_{n \to \infty} \|x_n - Tx_n\| = 0.$$ 

One question arises naturally: Is Theorem D with the conditions (ii) in Abstract above $\sum_{n=1}^{\infty} t_n s_n < \infty$ and $s_n \to 0$ as $n \to \infty$ or $s_n = s$ for all $n \geq 1$ and $s \in [0, 1)$ instead of the condition (ii) in Theorem D true?

It is our purpose in this paper to solve the above problem by proving the following result:

Theorem 1. Let $D$ be a nonempty subset of a normed linear space $X$ and $T : D \rightarrow X$ be a nonexpansive mapping. Let $\{x_n\}$ and $\{t_n\}$, $\{s_n\}$ be a sequence in $D$ and sequences of real numbers, respectively, such that

(i) $0 \leq t_n \leq t < 1$ and $\sum_{n=1}^{\infty} t_n = \infty$,
(ii) (a) $0 \leq s_n \leq 1$, $\lim_{n \to \infty} s_n = 0$ and $\sum_{n=1}^{\infty} t_n s_n < \infty$ or (b) $s_n = s$ for all $n \geq 1$ and $s \in [0, 1)$,
(iii) $x_{n+1} = (1-t_n)x_n + t_n T(s_n Tx_n + (1-s_n)x_n)$ for all $n \geq 1$. 

If \( \{ x_n \} \) is bounded, then
\[
\lim_{n \to \infty} \| x_n - Tx_n \| = 0.
\]

In order to prove Theorem 1, we shall need several important lemmas.

**Lemma 2.** ([1]) Let \( \{ a_n \} \), \( \{ b_n \} \) and \( \{ \delta_n \} \) be sequences of nonnegative real numbers satisfying the inequality
\[
a_{n+1} \leq (1 + \delta_n) a_n + b_n.
\]
If \( \sum_{n=1}^{\infty} \delta_n < \infty \) and \( \sum_{n=1}^{\infty} b_n < \infty \), then \( \lim_{n \to \infty} a_n \) exists. In particular, if \( \{ a_n \} \) has a subsequence converging to zero, then \( \lim_{n \to \infty} a_n = 0 \).

**Lemma 3.** Let \( X \) be a normed linear space and \( D \) be a nonempty subset of \( X \). Let \( T : D \to X \) be a nonexpansive mapping. Let \( \{ x_n \} \) be a sequence in \( D \) satisfying the condition (iii) in Theorem 1. Then we have
\[
\| x_{n+1} - Tx_{n+1} \| \leq (1 + 2t_n s_n) \| x_n - Tx_n \|
\]
for all \( n \geq 1 \).

**Proof.** Observe first that
\[
\| x_n - Ty_n \| \leq \| x_n - Tx_n \| + \| Tx_n - Ty_n \|
\leq \| x_n - Tx_n \| + \| x_n - y_n \|
= \| x_n - Tx_n \| + s_n \| x_n - Tx_n \|
= (1 + s_n) \| x_n - Tx_n \|
\]
and
\[
\| x_{n+1} - x_n \| = t_n \| x_n - Ty_n \|
\leq t_n (1 + s_n) \| x_n - Tx_n \|.
\]
Thus it follows that
\[
\| x_{n+1} - Tx_{n+1} \|
\leq (1 - t_n) \| x_n - Tx_{n+1} \| + t_n \| Ty_n - Tx_{n+1} \|
\leq (1 - t_n) (\| x_n - Tx_n \| + \| Tx_n - Tx_{n+1} \|) + t_n \| y_n - x_{n+1} \|
\leq (1 - t_n) \| x_n - Tx_n \| + (1 - t_n) \| x_n - x_{n+1} \|
+ t_n \| y_n - x_n \| + t_n \| x_n - x_{n+1} \|
\leq (1 - t_n + t_n s_n) \| x_n - Tx_n \| + t_n (1 + s_n) \| x_n - Tx_n \|
= (1 + 2t_n s_n) \| x_n - Tx_n \|.
This completes the proof.

Recall that a metric space \((X, d)\) is said to be of hyperbolic type if \(X\) contains a family \(L\) of metric segments such that

(a) each two points \(x, y \in X\) are endpoints of exactly one member segment of \(L\),

(b) if \(p, x, y \in X\) and \(m \in \text{seg}[x, y]\) satisfies \(d(x, m) = \alpha d(x, y)\) for \(\alpha \in [0, 1]\), then

\[
d(p, m) \leq (1 - \alpha)d(p, x) + \alpha d(p, y).
\]

This class includes all normed linear spaces as well as all spaces with the metric of hyperbolic type (see [3]).

**Lemma 4.** ([3]) Let \((X, d)\) be of hyperbolic type and \(\{t_n\}\) be a sequence in \([0, 1)\). Let \(\{x_n\}\) and \(\{y_n\}\) be sequences in \(X\) such that, for all \(n \geq 1\),

(i) \(x_{n+1} \in \text{seg}[x_n, y_n]\) with \(d(x_n, x_{n+1}) = t_n d(x_n, y_n)\),

(ii) \(d(y_{n+1}, y_n) \leq d(x_{n+1}, x_n)\),

(iii) \(d(y_{i+n}, x_i) \leq M < \infty\) for all \(i, n \geq 1\),

(iv) \(t_n \leq t < 1\),

(v) \(\sum_{n=1}^{\infty} t_n = \infty\).

Then we have

\[
\lim_{n \to \infty} d(y_n, x_n) = 0.
\]

The proof of Theorem 1. We assume first that \(\sum_{n=1}^{\infty} t_n s_n < \infty\) and \(s_n \to 0\) as \(n \to \infty\). By Lemmas 2 and 3, we see that \(\|x_n - Tx_n\|\) exists, say \(d\). Setting \(a_n = Tx_n - x_n\), then we have \(\|a_n\| \to d\) as \(n \to \infty\). Without of loss of generality, we may assume that \(t_n > 0\) for all \(n \geq 1\). Otherwise, consider a subsequence \(\{t_j\}\) of \(\{t_n\}\). Setting

\[
b_n = t_n^{-1}(Tx_{n+1} - Tx_n) + Tx_n - Ty_n,
\]

we have \(a_{n+1} = (1 - t_n)a_n + t_nb_n\). Following the proof lines of Deng [2, Theorem 1], we can get the following conclusions:

1. \(\limsup_{n \to \infty} \|b_n\| \leq d\),
2. \(\|\sum_{i=1}^{n} t_i b_i\| \leq \|x_{n+1} - x_1\| + \sum_{i=1}^{n} t_i s_i \|Tx_i - x_i\|\) is bounded.

Thus the conclusion of Theorem 1 follows exactly from Deng [2, Lemma 2].
Next, we assume that $s_n = s$ for all $n \geq 1$ and $s \in [0, 1)$. In order to end the proof of Theorem 1, we only need to verify the conditions (i)\textasciitilde(v) in Lemma 4.

The condition (i) is obvious and the conditions (iv), (v) are natural. Now, we prove only the conditions (ii), (iii).

Setting $z_n = (1 - s)x_n + sTx_n$ and $y_n = Tz_n$, then we have

$$
d(y_{n+1}, y_n) = \|y_{n+1} - y_n\| = \|Tz_{n+1} - Tz_n\| \\
\leq \|z_{n+1} - z_n\| = \|(1 - s)x_{n+1} + sTx_{n+1} - (1 - s)x_n - sTx_n\| \\
\leq (1 - s)\|x_{n+1} - x_n\| + s\|Tx_{n+1} - Tx_n\| \\
\leq (1 - s)\|x_{n+1} - x_n\| + s\|x_{n+1} - x_n\| \\
= \|x_{n+1} - x_n\| = d(x_{n+1}, x_n),
$$

which verifies the condition (ii).

Next, we have the following.

$$
d(y_{i+n}, x_i) = \|y_{i+n} - x_i\| = \|Tz_{i+n} - x_i\| \\
\leq \|Tz_{i+n} - Tx_{i+n}\| + \|Tx_{i+n} - x_i\| \\
\leq \|z_{i+n} - x_{i+n}\| + \|Tx_{i+n} - x_i\| \\
\leq s\|x_{i+n} - Tx_{i+n}\| + \|Tx_{i+n} - x_i\| \\
\leq M < \infty,
$$

which verifies the condition (iii).

By Lemma 4, we assert that $\lim_{n \to \infty} \|Tz_n - x_n\| = 0$. Observe that

$$
\|Tx_n - x_n\| \leq \|Tx_n - Tz_n\| + \|Tz_n - x_n\| \\
\leq \|x_n - z_n\| + \|Tz_n - x_n\| \\
= s\|x_n - Tx_n\| + \|Tz_n - x_n\|,
$$

which implies that

$$
\lim_{n \to \infty} \|x_n - Tx_n\| = 0.
$$

This completes the proof. □
As immediate consequences of Theorem 1, we have the following.

**Corollary 5.** Let $X$ be a real normed linear space, $D$ be a nonempty compact convex subset of $X$ and $T : D \to D$ be a nonexpansive mapping. Let the sequence $\{x_n\}$ in $D$ be defined as in Theorem 1. Then $\{x_n\}$ converges strongly to a fixed point of $T$.

**Proof.** Since $D$ is a convex subset of $X$ and $T : D \to D$ is a self-mapping, we see that the sequence $\{x_n\}$ is well-defined. From the compactness of $D$, we assert that there exists some subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to p$ as $j \to \infty$. Thus it follows from Theorem 1 that $Tx_{n_j} \to p$ as $j \to \infty$. By virtue of the continuity of $T$, we conclude that $Tp = p$, which means that $p \in F(T)$, where $F(T)$ denotes the set of fixed points of $T$. Since $T : D \to D$ is nonexpansive, we have

$$\|x_{n+1} - q\| \leq \|x_n - q\|$$

for all $n \geq 1$ and $q \in F(T)$ (see [6]), which shows that $\lim_{n \to \infty} \|x_n - q\|$ exists for all $q \in F(T)$. Consequently, we have $x_n \to p$ as $n \to \infty$. This completes the proof. $\square$

Recall that a mapping $T : D \to D$ with the fixed point set $F(T)$ satisfies Condition (A) if there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\|x - Tx\| \geq \phi(d(x, F(T)))$$

for all $x \in D$ (see [5]).

**Corollary 6.** Let $X$ be a real Banach space, $D$ be a nonempty closed convex subset of $X$ and $T : D \to D$ be a nonexpansive mapping with the nonempty fixed point set $F(T)$. Let the sequence $\{x_n\}$ in $D$ be as in Theorem 1. If the mapping $T$ satisfies Condition (A), then $\{x_n\}$ converges strongly to a fixed point of $T$.

**Proof.** It follows from Theorem 1 and Condition (A) that

$$\phi(d(x_n, F(T))) \to 0$$

as $n \to \infty$. Since $\phi : [0, \infty) \to [0, \infty)$ is strictly increasing, we assert that $d(x_n, F(T)) \to 0$ as $n \to \infty$. Observe that

$$\|x_{n+m} - q\| \leq \|x_n - q\|$$
for all $n, m \geq 1$ and $q \in F(T)$, which implies that
\[
\|x_{n+m} - x_n\| \leq \|x_{n+m} - q\| + \|x_n - q\| \\
\leq 2\|x_n - q\|
\]
and hence
\[
\|x_{n+m} - x_n\| \leq 2d(x_n, F(T)),
\]
which shows that $\{x_n\}$ is a Cauchy sequence in $D$. Since $D$ is complete, we may assume that $x_n \to p$ as $n \to \infty$. Therefore, $d(p, F(T)) = 0$ and $p \in F(T)$ because of the closedness of $F(T)$. This completes the proof. □

**Remark 1.** In view of Theorem 1, we can also establish several weak and strong convergence theorems similar to Theorems 2~6 of Deng [2]. We omit to prove them because of similarity of the proof lines.

**Remark 2.** For the parameters of our theorems, one can make the following choices: If we put $t_n = \frac{1}{n+1}$ and $s_n = \frac{1}{n}$ or $s_n = s \in [0, 1)$ for all $n \geq 1$. Then these parameters satisfy all requirements of our results, however, they do not satisfy the requirements of Theorem D and the results of Tan and Xu [6].

Recall that a Banach space $X$ satisfies Opial’s condition if the condition $x_n \to x_0$ weakly implies
\[
\limsup_{n \to \infty} \|x_n - x_0\| < \limsup_{n \to \infty} \|x_n - y\|
\]
for all $y \neq x_0$.

**Theorem 7.** Let $X$ be a Banach space which satisfies Opial’s condition, $D$ be weakly compact and $T$, $\{x_n\}$ be as in Theorem 1. Then $\{x_n\}$ converges weakly to a fixed point of $T$.

**Proof.** By the weak compactness of $D$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to a point $p \in D$. With the standard proof, we can show that $p = Tp$. Suppose that $\{x_n\}$ does not converge weakly to the point $p$. Then there exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $p \neq q$ such that $x_{n_j} \to q$ weakly and $q = Tq$. Since $T : D \to X$ is nonexpansive, we have
\[
\|x_{n+1} - p\| \leq \|x_n - p\|
\]
for all $n \geq 1$ and $p \in F(T)$ (see [6]). Thus, by Opial’s condition of $X$,

$$\lim_{n \to \infty} \|x_n - p\| = \lim_{k \to \infty} \|x_{n_k} - p\| < \lim_{k \to \infty} \|x_{n_k} - q\|$$

$$= \lim_{j \to \infty} \|x_{n_j} - q\| < \lim_{j \to \infty} \|x_{n_j} - p\|$$

$$= \lim_{n \to \infty} \|x_n - p\|,$$

which is a contradiction. This completes the proof. □

References


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