(WEAK) IMPLICATIVE HYPER $K$-IDEALS

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Abstract. In this note first we define the notions of weak implicative and implicative hyper $K$-ideals of a hyper $K$-algebra $H$. Then we state and prove some theorems which determine the relationship between these notions and (weak) hyper $K$-ideals. Also we give some relations between these notions and all types of positive implicative hyper $K$-ideals. Finally we classify the implicative hyper $K$-ideals of a hyper $K$-algebra of order 3.

1. Introduction

The hyperalgebraic structure theory was introduced by F. Marty [9] in 1934. Imai and Iseki [5] in 1966 introduced the notion of a $BCK$-algebra. Recently [2, 3, 12] Borzooei, Jun and Zahedi et.al. applied the hyperstructure to $BCK$-algebras and introduced the concept of hyper $K$-algebra which is a generalization of $BCK$-algebra. Now, in this note we define the notions of (weak) implicative hyper $K$-ideals, then we obtain some related results which have been mentioned in the abstract.

2. Preliminaries

Definition 2.1. [2] Let $H$ be a nonempty set and “$\circ$” be a hyperoperation on $H$, that is “$\circ$” is a function from $H \times H$ to $\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\emptyset\}$. Then $H$ is called a hyper $K$-algebra if it contains a constant “0” and satisfies the following axioms:

(HK1) $(x \circ z) \circ (y \circ z) < x \circ y$
(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$
(HK3) $x < x$
(HK4) $x < y$, $y < x \Rightarrow x = y$

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for all \( x, y, z \in H \), where \( x < y \) is defined by \( 0 \in x \circ y \) and for every \( A, B \subseteq H \), \( A < B \) is defined by \( \exists a \in A, \exists b \in B \) such that \( a < b \).

Note that if \( A, B \subseteq H \), then by \( A \circ B \) we mean the subset
\[
\bigcup_{a \in A, b \in B} a \circ b
\]
of \( H \).

**Example 2.2.** [2] Define the hyperoperation “\( \circ \)” on \( H = [0, +\infty) \) as follows:
\[
x \circ y = \begin{cases} 
[0, x] & \text{if } x \leq y \\
(0, y] & \text{if } x > y \\
\{x\} & \text{if } y = 0
\end{cases}
\]
for all \( x, y \in H \). Then \((H, \circ, 0)\) is a hyper \( K \)-algebra.

**Theorem 2.3.** [2] Let \((H, \circ, 0)\) be a hyper \( K \)-algebra. Then for all \( x, y, z \in H \) and for all nonempty subsets \( A, B \) and \( C \) of \( H \) the following hold:

(i) \( x \circ y < z \iff x \circ z < y \),
(ii) \( (x \circ z) \circ (x \circ y) < y \circ z \),
(iii) \( x \circ (x \circ y) < y \),
(iv) \( x \circ y < x \),
(v) \( A \subseteq B \) implies \( A < B \),
(vi) \( x \in x \circ 0 \),
(vii) \( (A \circ C) \circ (A \circ B) < B \circ C \),
(viii) \( (A \circ C) \circ (B \circ C) < A \circ B \),
(ix) \( A \circ B < C \iff A \circ C < B \).

**Definition 2.4.** [2] Let \( I \) be a nonempty subset of a hyper \( K \)-algebra \((H, \circ, 0)\) and \( 0 \in I \). Then,

(i) \( I \) is called a **weak hyper \( K \)-ideal** of \( H \) if \( x \circ y \subseteq I \) and \( y \in I \) imply that \( x \in I \) for all \( x, y \in H \).

(ii) \( I \) is called a **hyper \( K \)-ideal** of \( H \) if \( x \circ y < I \) and \( y \in I \) imply that \( x \in I \) for all \( x, y \in H \).

**Theorem 2.5.** [2] Any hyper \( K \)-ideal of a hyper \( K \)-algebra \( H \), is a weak hyper \( K \)-ideal.

**Definition 2.6.** [1] Let \( I \) be a nonempty subset of a hyper \( K \)-algebra \((H, \circ, 0)\) such that \( 0 \in I \). Then \( I \) is called a **positive implicative hyper \( K \)-ideal of**

(i) **type 1**, if for all \( x, y, z \in H \), \( (x \circ y) \circ z \subseteq I \) and \( y \circ z \subseteq I \) imply that
(Weak) implicative hyper $K$-ideals

$$x \circ z \subseteq I,$$

(ii) **type** 2, if for all $x, y, z \in H$, $(x \circ y) \circ z < I$ and $y \circ z \subseteq I$ imply that $x \circ z \subseteq I$,

(iii) **type** 3, if for all $x, y, z \in H$, $(x \circ y) \circ z < I$ and $y \circ z < I$ imply that $x \circ z \subseteq I$,

(iv) **type** 4, if for all $x, y, z \in H$, $(x \circ y) \circ z \subseteq I$ and $y \circ z < I$ imply that $x \circ z \subseteq I$,

(v) **type** 5, if for all $x, y, z \in H$, $(x \circ y) \circ z \subseteq I$ and $y \circ z \subseteq I$ imply that $x \circ z < I$,

(vi) **type** 6, if for all $x, y, z \in H$, $(x \circ y) \circ z < I$ and $y \circ z < I$ imply that $x \circ z < I$,

(vii) **type** 7, if for all $x, y, z \in H$, $(x \circ y) \circ z \subseteq I$ and $y \circ z < I$ imply that $x \circ z < I$,

(viii) **type** 8, if for all $x, y, z \in H$, $(x \circ y) \circ z < I$ and $y \circ z \subseteq I$ imply that $x \circ z < I$.

**Definition 2.7.** [3] Let $I$ be a nonempty subset of $H$. Then we say that $I$ satisfies the **additive condition** if for all $x, y \in H$, $x < y$ and $y \in I$ imply that $x \in I$.

**Definition 2.8.** [1] Let $H$ be a hyper $K$-algebra. An element $a \in H$ is called a **left** (resp. **right**) scalar if $|a \circ x| = 1$ (resp. $|x \circ a| = 1$) for all $x \in H$. If $a \in H$ is both left and right scalar, we say that $a$ is an **scalar** element.

**Definition 2.9.** [1] We say that the hyper $K$-algebra $H$ satisfies the **transitive condition** if for all $x, y, z \in H$, $x < y$ and $y < z$ imply that $x < z$.

3. Some results on hyper $K$-ideals

From now on $H$ is a hyper $K$-algebra, unless otherwise is stated.

**Proposition 3.1.** Let $I$ be a hyper $K$-ideal of $H$, and $A, B \subseteq H$. If $A \circ B < I$ and $B \subseteq I$, then $A < I$. 
Proof. We have $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$ and $A \circ B < I$. Thus there exist $t \in a \circ b$ for some $a \in A$, $b \in B$ and $s \in I$ such that $t < s$. Hence $a \circ b < I$. Since $I$ is a hyper $K$-ideal and $b \in I$ we conclude that $a \in I$, thus $A < I$.

**Remark 3.2.** (i) In the above proposition it is not necessary that $A \subseteq I$. To show this, let $H = \{0, 1, 2\}$. Then the following table shows a hyper $K$-algebra structure on $H$.

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Now, $I = \{0, 1\}$ is a hyper $K$-ideal of $H$, $\{1, 2\} \circ \{0, 1\} = \{0, 1, 2\} < I$ and $\{0, 1\} \subseteq I$, but $\{1, 2\} \not\subseteq I$.

(ii) If in Proposition 3.1, we use $B < I$ instead of $B \subseteq I$, then the result does not hold. Because consider $H = \{0, 1, 2\}$, then the following table shows a hyper $K$-algebra structure on $H$.

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Let $I = \{0\}$, clearly $I$ is a hyper $K$-ideal. We have $\{1\} \circ \{0, 1, 2\} < I$ and $\{0, 1, 2\} < I$, but $\{1\} \not< I$.

**Lemma 3.3.** Let $I$ be a weak hyper $K$-ideal of $H$. If for all $A, B \subseteq H$, $A \circ B \subseteq I$ and $B \subseteq I$, then $A \subseteq I$.

**Proof.** For all $a \in A$, $b \in B$ we have $a \circ b \subseteq A \circ B \subseteq I$ and $b \in I$. Since $I$ is a weak hyper $K$-ideal, we get that $a \in I$, thus $A \subseteq I$.

**Remark 3.4.** In the above lemma the condition $B \subseteq I$ can not be replaced by $B < I$. Because let $H = \{0, 1, 2\}$. Then the following table...
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shows a hyper $K$-algebra structure on $H$.

\[
\begin{array}{c|ccc}
\circ & 0 & 1 & 2 \\
\hline
0 & \{0\} & \{0\} & \{0\} \\
1 & \{1\} & \{0, 1, 2\} & \{2\} \\
2 & \{2\} & \{0, 1, 2\} & \{0, 1\}
\end{array}
\]

Now, $I = \{0, 1\}$ is a weak hyper $K$-ideal of $H$, $2 \circ (1 \circ 2) \subseteq I$ and $1 \circ 2 < I$, while $\{2\} \not\subseteq I$.

**Definition 3.5.** We say that $H$ satisfies the **strong transitive condition** if for all $A, B, C \subseteq H$, $A < B$ and $B < C$ imply that $A < C$.

**Corollary 3.6.** Let $H$ satisfies the strong transitive condition. Then it satisfies the transitive condition.

**Proof.** It is easy.

The following example shows that the converse of the above corollary is not true in general. To show this let $H = \{0, 1, 2\}$. Then the following table shows a hyper $K$-algebra structure on $H$.

\[
\begin{array}{c|ccc}
\circ & 0 & 1 & 2 \\
\hline
0 & \{0\} & \{0\} & \{0\} \\
1 & \{1\} & \{0\} & \{1\} \\
2 & \{2\} & \{2\} & \{0, 1\}
\end{array}
\]

It is easy to check that $H$ satisfies the transitive condition, while it does not satisfy the strong transitive condition. Because $\{2\} < \{1, 2\}$ and $\{1, 2\} < \{1\}$, but $\{2\} \not< \{1\}$.

**Proposition 3.7.** Let $H$ satisfies the strong transitive condition. If $I$ is a hyper $K$-ideal of $H$ and $A, B \subseteq H$, $A \circ B < I$ and $B < I$, then $A < I$.

**Proof.** Let $A \circ B < I$. Then by Theorem 2.3 (ix) we have $A \circ I < B$, and $B < I$. Since $H$ satisfies the strong transitive condition we get that $A \circ I < I$. Now by Proposition 3.1 we have $A < I$. \(\Box\)
4. Implicative hyper $K$-ideal

Definition 4.1. A nonempty subset $I$ of $H$ is called a weak implicative hyper $K$-ideal if it satisfies:

(i) $0 \in I$
(ii) $(x \circ z) \circ (y \circ x) \subseteq I$ and $z \in I$ imply $x \in I$, for all $x, y, z \in H$.

Example 4.2. Let $H = \{0, 1, 2\}$. Then the following table shows a hyper $K$-algebra structure on $H$.

\[
\begin{array}{c|ccc}
\circ & 0 & 1 & 2 \\
\hline
0 & \{0\} & \{0\} & \{0\} \\
1 & \{1\} & \{0, 1\} & \{1\} \\
2 & \{1, 2\} & \{0, 1\} & \{0, 1\}
\end{array}
\]

Then $I = \{0, 2\}$ is a weak implicative hyper $K$-ideal of $H$.

Definition 4.3. A nonempty subset $I$ of $H$ is called an implicative hyper $K$-ideal if it satisfies:

(i) $0 \in I$
(ii) $(x \circ z) \circ (y \circ x) < I$ and $z \in I$ imply $x \in I$, for all $x, y, z \in H$.

Example 4.4. Let $H = \{0, 1, 2\}$. The following table shows a hyper $K$-algebra structure on $H$.

\[
\begin{array}{c|ccc}
\circ & 0 & 1 & 2 \\
\hline
0 & \{0\} & \{0\} & \{0\} \\
1 & \{1\} & \{0, 2\} & \{1\} \\
2 & \{2\} & \{0, 2\} & \{0, 2\}
\end{array}
\]

Then $I = \{0, 2\}$ is an implicative hyper $K$-ideal, while $I = \{0, 1\}$ is not an implicative hyper $K$-ideal, because $(2 \circ 0) \circ (1 \circ 2) < I$, and $0 \in I$ but $2 \notin I$.

Proposition 4.5. Each implicative hyper $K$-ideal of $H$ is a weak implicative.

Proof. Let $I$ be an implicative hyper $K$-ideal and $(x \circ z) \circ (y \circ x) \subseteq I$, $z \in I$. Then by Theorem 2.3 (v) we have $(x \circ z) \circ (y \circ x) < I$, thus $x \in I$. So $I$ is a weak implicative hyper $K$-ideal.

The following example shows that the converse of the above proposition is not correct in general. Consider $H = \{0, 1, 2\}$. The following
Then \( I = \{0, 1\} \) is a weak implicative hyper \( K \)-ideal, while it is not an implicative hyper \( K \)-ideal, because \((2 \circ 0) \circ (1 \circ 2) < I\), \(0 \in I\) but \(2 \notin I\).

**Theorem 4.6.** Every implicative hyper \( K \)-ideal of \( H \) is a hyper \( K \)-ideal.

**Proof.** Let \( I \) be an implicative hyper \( K \)-ideal of \( H \), \( x \circ y < I \) and \( y \in I \). Then there exist \( t \in x \circ y \) and \( z \in I \) such that \( t < z \). We have \( t \in t \circ 0 \subseteq (x \circ y) \circ (0 \circ x) \). Thus \((x \circ y) \circ (0 \circ x) < I \) and \( y \in I \), therefore \( x \in I \).

The following example shows that the converse of the above theorem is not correct in general. Let \( H = \{0, 1, 2\} \). Then the following table shows a hyper \( K \)-algebra structure on \( H \).

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Now, we can see that \( I = \{0, 1\} \) is a weak implicative hyper \( K \)-ideal, while it is not a weak hyper \( K \)-ideal, because \(2 \circ 1 \subseteq I\) and \(1 \in I\), but \(2 \notin I\).

**Remark 4.7.** (i) In general, a weak implicative hyper \( K \)-ideal does not need to be a weak hyper \( K \)-ideal. To show this, consider \( H = \{0, 1, 2\} \), then the following table shows a hyper \( K \)-algebra structure on \( H \).

<table>
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<td>{0, 1, 2}</td>
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</table>

We can check that \( I = \{0, 2\} \) is a hyper \( K \)-ideal, while it is not an implicative hyper \( K \)-ideal, since \((1 \circ 0) \circ (2 \circ 1) = \{0, 1, 2\} < I \) and \(0 \in I\), but \(1 \notin I\).

(ii) In general, a weak hyper \( K \)-ideal does not need to be a weak implicative hyper \( K \)-ideal. For this consider the hyper \( K \)-algebra \( H \) of Remark 3.4. Then \( I = \{0, 1\} \) is a weak hyper \( K \)-ideal, while it is not a
weak implicative hyper \( K \)-ideal, since \((2 \circ 0) \circ (1 \circ 2) \subseteq I\), and \(0 \in I\), but \(2 \notin I\).

**Theorem 4.8.** Let \(I\) be a weak hyper \( K \)-ideal of \(H\). Then the following statements hold:

(i) If for all \(x, y, z \in H\), \(x \circ (y \circ x) \subseteq I\) implies \(x \in I\), then \(I\) is a weak implicative hyper \( K \)-ideal.

(ii) Let \(0 \in H\) be a right scalar element and \(I\) be a weak implicative hyper \( K \)-ideal. Then for all \(x, y \in H\), \(x \circ (y \circ x) \subseteq I\), implies that \(x \in I\).

**Proof.** (i) Let \(I\) be a weak hyper \( K \)-ideal, \((x \circ z) \circ (y \circ x) \subseteq I\) and \(z \in I\). Then \((x \circ (y \circ x)) \circ z \subseteq I\). By Lemma 3.3, we have \(x \circ (y \circ x) \subseteq I\). Now by hypothesis \(x \in I\). So \(I\) is a weak implicative hyper \( K \)-ideal.

(ii) Let \(I\) be a weak implicative hyper \( K \)-ideal, \(x \circ (y \circ x) \subseteq I\) and \(0 \in H\) is a right scalar element. We have \((x \circ 0) \circ (y \circ x) = x \circ (y \circ x) \subseteq I\) and \(0 \in I\), thus \(x \in I\). \(\square\)

The following theorem shows that if we restrict to a hyper \( K \)-algebra of order 3, then we can omit the condition “\(0 \in H\) be a right scalar element”, in the above theorem.

**Theorem 4.9.** Let \(H = \{0, 1, 2\}\) be a hyper \( K \)-algebra of order 3, and \(I\) be a proper weak hyper \( K \)-ideal of \(H\). Then \(I\) is a weak implicative hyper \( K \)-ideal if and only if for all \(x, y \in H\), \(x \circ (y \circ x) \subseteq I\) implies \(x \in I\).

**Proof.** Let \(I = \{0, 1\}\) be a proper weak hyper \( K \)-ideal and also a weak implicative hyper \( K \)-ideal of \(H\). If \(x \circ (y \circ x) \subseteq I\), for arbitrary elements \(x, y \in H\), then we show that \(x \in I\). If \(x = 0\) or \(1\), then it is done. So let \(x = 2\), therefore
\[
(1) \quad 2 \circ (y \circ 2) \subseteq I.
\]

We know that \(0 \notin 2 \circ 0\) and \(2 \in 2 \circ 0\). Thus \(2 \circ 0 = \{2\}\) or \(2 \circ 0 = \{1, 2\}\). If \(2 \circ 0 = \{2\}\), then \((2 \circ 0) \circ (y \circ 2) = 2 \circ (y \circ 2) \subseteq I\), by (1). Since \(0 \in I\) and \(I\) is a weak implicative hyper \( K \)-ideal, we get that \(2 \in I\), which is a contradiction.

If \(2 \circ 0 = \{1, 2\}\), then we consider the following different cases.

(i) If \(y = 0\), then \(2 \in 2 \circ 0 \subseteq 2 \circ (0 \circ 2) \subseteq I\), by (1), which is a contradiction.

(ii) If \(y = 1\) and \(1 < 2\), then \(0 \in 1 \circ 2\). Thus \(2 \in 2 \circ 0 \subseteq 2 \circ (1 \circ 2) \subseteq I\), by (1). Which is a contradiction.
If $y = 1$ and $1 \not< 2$, then $1 \circ 2 = \{1\}$ or $\{1, 2\}$ or $\{2\}$. So we must discuss on the above different cases:

(a) If $1 \circ 2 = \{1\}$, then $2 \circ 1 = 2 \circ (1 \circ 2) \subseteq I$, by (1). Since $1 \in I$ and $I$ is a weak hyper $K$-ideal, we conclude that $2 \in I$, which is a contradiction.

(b) If $1 \circ 2 = \{1, 2\}$, then $(2 \circ 1) \cup (2 \circ 2) = 2 \circ \{1, 2\} = 2 \circ (1 \circ 2) \subseteq I$, by (1). Hence $2 \circ 1 \subseteq I$. Therefore $2 \in I$, which is a contradiction.

(c) If $1 \circ 2 = \{2\}$, then we claim that $1 \circ 0 = \{1\}$. Suppose $1 \circ 0 \neq \{1\}$. Since $1 \in 1 \circ 0$ and $0 \not\in 1 \circ 0$, we must have $1 \circ 0 = \{1, 2\}$. Then $0 \in 2 \circ 0 \subseteq \{2\} \cup 2 \circ 2 = 1 \circ 2 \cup 2 \circ 2 = \{1, 2\} \circ 2 = (1 \circ 0) \circ 2$, so

\[0 \in (1 \circ 0) \circ 2.\]

On the other hand $(1 \circ 0) \circ 2 = (1 \circ 2) \circ 0 = 2 \circ 0$. Since $0 \not\in 2 \circ 0$, we get that $0 \not\in (1 \circ 0) \circ 2$, which is a contradiction by (2). Thus we must have $1 \circ 0 = \{1\}$. Therefore

\[(1 \circ 2) \circ 0 = 2 \circ 0 = \{1, 2\}\]

and

\[(1 \circ 0) \circ 2 = 1 \circ 2 = \{2\}.\]

Since $(1 \circ 2) \circ 0 = (1 \circ 0) \circ 2$. So (3), (4) given a contradiction. Thus $y = 1$ does not happen.

(iii) Let $y = 2$. Then $2 \in 2 \circ 0 \subseteq 2 \circ (2 \circ 2) \subseteq I$, by (1). Which is a contradiction. Therefore the above argument shows that $x \neq 2$, i.e., $x \in I$. Finally by considering Theorem 4.8, the proof of the converse is obvious.

**Definition 4.10.** [11] Let $H = \{0, 1, 2\}$ be a hyper $K$-algebra of order 3. We say that $H$ satisfies the simple condition if $1 \not< 2$ and $2 \not< 1$.

**Theorem 4.11.** Let $H = \{0, 1, 2\}$ be a hyper $K$-algebra of order 3, that satisfies the simple condition, and let $\{0\} \neq I \subset H$. Then $I$ is a weak hyper $K$-ideal of $H$ if and only if $I$ is a weak implicative $K$-ideal of $H$.

**Proof.** Let $I$ be a weak hyper $K$-ideal of $H$. By hypothesis we have $I = \{0, 1\}$ or $\{0, 2\}$. Let $I = \{0, 1\}$. By Theorem 4.9 it is enough to show that if $x \circ (y \circ x) \subseteq I$, for any two arbitrary elements $x, y$ of $H$, then $x \in I$. So let $x \circ (y \circ x) \subseteq I$. If $x = 0$ or 1, then it is done. Thus let $x = 2$. Consider the following different cases:

**Case (1).** If $y = 0$, then $2 \in 2 \circ 0 \subseteq 2 \circ (2 \circ 0) \subseteq I$ and hence $2 \in I$, which is a contradiction.
Case (2). If \( y = 1 \), since \( H \) satisfies the simple condition then \( 1 \not< 2 \) and \( 0 \not< 1 \circ 2 \). Hence \( 1 \circ 2 = \{1\}, \{2\} \) or \( \{1, 2\} \).

(i) If \( 1 \circ 2 = \{1\} \), then \( 2 \circ 1 = 2 \circ (1 \circ 2) \subseteq I \). Since \( I \) is a weak hyper \( K \)-ideal and \( 1 \in I \) then we get that \( 2 \in I \), which is a contradiction.

(ii) The case \( 1 \circ 2 = \{2\} \) does not happen, by Theorem 3.17 of [11].

(iii) If \( 1 \circ 2 = \{1, 2\} \), then \( (2 \circ 1) \cup (2 \circ 2) = 2 \circ \{1, 2\} = 2 \circ (1 \circ 2) \subseteq I \). Thus \( 2 \circ 1 \subseteq I \). Now \( 1 \in I \) implies that \( 2 \in I \), which is a contradiction.

Case (3). If \( y = 2 \), then \( 2 \in 2 \circ 0 \subseteq 2 \circ (2 \circ 2) \subseteq I \). Hence \( 2 \in I \), which is a contradiction.

Thus \( x \neq 2 \). Hence \( x \in I \). Note that the proof of the case \( I = \{0, 2\} \) is similar as above.

Conversely, let \( I \) be a weak implicative hyper \( K \)-ideal of \( H \). Without loss of generality we assume that \( I = \{0, 1\} \). Let \( x \circ y \subseteq I \) and \( y \in I \). If \( x = 0 \) or \( 1 \), then \( x \in I \). So let \( x = 2 \). We consider the following cases:

Case (1). The case \( y = 0 \) does not happen, because \( 2 = 2 \circ 0 \not\subseteq I \).

Case (2). If \( y = 1 \), since \( 2 \not< 1 \), then \( 0 \not< 2 \circ 1 \). Hence \( 2 \circ 1 = \{1\}, \{2\} \) or \( \{1, 2\} \). Since \( H \) satisfies the simple condition, then by Theorem 3.17 of [11] \( 2 \circ 1 \neq \{1\} \). So the cases \( 2 \circ 1 = \{2\} \) or \( \{1, 2\} \) do not happen, since \( 2 \circ 1 \not\subseteq I \).

Case (3). The case of \( y = 2 \) does not happen, because \( 2 \not\in I \).

Consequently \( x \neq 2 \), hence \( x \circ y \subseteq I \) and \( y \in I \) imply that \( x \in I \), for all \( x, y \in H \). This shows that \( I \) is a weak implicative hyper \( K \)-ideal. Note that the proof of the case \( I = \{0, 2\} \) is similar as above.

**Theorem 4.12.** Let \( I \) be a hyper \( K \)-ideal of \( H \). Then \( I \) is an implicative hyper \( K \)-ideal if and only if

\[
(5) \quad x \circ (y \circ x) \not< I \quad \text{implies that} \quad x \in I, \quad \text{for any} \quad x, y \in H.
\]

**Proof.** Let \( I \) satisfies in (5) and \( (x \circ z) \circ (y \circ x) \not< I \), \( z \in I \). Then by Proposition 3.1 we have \( x \circ (y \circ x) \not< I \). So by (5) we get that \( x \in I \). Therefore \( I \) is an implicative hyper \( K \)-ideal.

Conversely, let \( I \) be an implicative hyper \( K \)-ideal, and \( x \circ (y \circ x) \not< I \). Since \( x \circ (y \circ x) \subseteq (x \circ 0) \circ (y \circ x) \), we conclude that \( (x \circ 0) \circ (y \circ x) \not< I \). Thus \( 0 \in I \) implies that \( x \in I \).

**Theorem 4.13.** Let \( H \) satisfies the strong transitive condition. If \( I \) is an implicative hyper \( K \)-ideal of \( H \), then \( I \) is a positive implicative hyper \( K \)-ideal of types 1-8.

**Proof.** By considering Theorem 3.5 of [1], it is enough to show that \( I \) is a positive implicative hyper \( K \)-ideal of type 3. Let \( (x \circ y) \circ z \not< I \),
and \( y \circ z < I \), we must show that \( x \circ z \subseteq I \). Let \( t \in x \circ z \). Then by (HK1) we have
\[
(t \circ z) \circ (y \circ z) < t \circ y \subseteq (x \circ z) \circ y = (x \circ y) \circ z < I.
\]
Since \( H \) satisfies the strong transitive condition, then \((t \circ z) \circ (y \circ z) < I\). Since \( y \circ z < I \) by Proposition 3.7, we conclude that \( t \circ z < I \). Now, by Theorem 2.3 (ii) we have \((x \circ z) \circ (x \circ t) < t \circ z\), thus by hypothesis we get that \((x \circ z) \circ (x \circ t) < I\). Since \((x \circ z) \circ (x \circ t) \subseteq (x \circ z) \circ (x \circ (x \circ z))\), we conclude that \((x \circ z) \circ (x \circ (x \circ z)) < I\). But for all \( t \in x \circ z \) we have \( t \circ (x \circ t) \subseteq (x \circ z) \circ (x \circ (x \circ z))\), so by hypothesis \( t \circ (x \circ t) < I \). Thus by Theorem 4.12, \( t \in I \), and hence \( x \circ z \subseteq I \).

\textbf{Remark 4.14}. In Theorem 4.13 the condition strong transitivity of \( H \) is essential. Because, let \( H = \{0, 1, 2\} \). Then the following table shows a hyper \( K \)-algebra structure on \( H \).

<table>
<thead>
<tr>
<th>( \circ )</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>{0}</td>
<td>{0}</td>
<td>{0}</td>
</tr>
<tr>
<td>1</td>
<td>{1}</td>
<td>{0}</td>
<td>{1}</td>
</tr>
<tr>
<td>2</td>
<td>{1, 2}</td>
<td>{0}</td>
<td>{0, 1}</td>
</tr>
</tbody>
</table>

Now \( H \) does not satisfy the strong transitive condition, because \( \{1\} < \{1, 2\} < \{2\} \) and \( \{1\} \not\subset \{2\} \). Clearly \( I = \{0, 2\} \) is an implicative hyper \( K \)-ideal of \( H \), but it is not a positive implicative hyper \( K \)-ideal of type 2 or 3. Because \((2 \circ 0) \circ 0 < I\) and \(0 \circ 0 \subseteq I\), but \(2 \circ 0 \not\subseteq I\).

\textbf{Theorem 4.15}. Let \( H = \{0, 1, 2\} \) be a hyper \( K \)-algebra of order 3, that satisfies the simple condition, and \( \{0\} \not\subset I \subset H \). Then \( I \) is an implicative hyper \( K \)-ideal if and only if \( I \) is a positive implicative hyper \( K \)-ideal of type 3.

\textit{Proof}. Let \( I \) be a positive implicative hyper \( K \)-ideal of type 3. Without loss of generality assume that \( I = \{0, 1\} \). Let \((x \circ z) \circ (y \circ x) < I\) and \( z \in I \), we show that \( z \in I \). By Theorems 17.3 and 19.3 of [11], we have \( 2 \circ 1 = \{2\}, 2 \circ 0 = \{2\}, 1 \circ 2 = \{1\}, 1 \circ 0 = \{1\}, x \circ y \neq \{0, 2\} \) and \( x \circ y \neq \{0, 1, 2\} \) for all \( x, y \in H \). Thus
\[
x \circ y \subseteq \{0, 1\}, \text{ for all } x, y \in H.
\]
Now, let \( x = 2 \). In the following we show that, this case is impossible. To this end consider three different cases:

(i) Let \( z = 0 \). We consider the following subcases:
(a) If \( y = 0 \), then by (6) we have \( 0 \circ 2 \subseteq \{0, 1\} \). Hence \((2 \circ 0) \circ (0 \circ 2) = 2 \circ (0 \circ 2) \subseteq 2 \circ \{0, 1\} = (2 \circ 0) \cup \{2\} = \{2\} \cup \{2\} = \{2\} \).

So by hypothesis \((2 \circ 0) \circ (0 \circ 2) < \{0, 1\}\), therefore \(\{2\} < \{0, 1\}\), which implies that \(2 < 1\). Thus we obtain a contradiction, because \(H\) satisfies the simple condition.

(b) If \( y = 1 \), then \((2 \circ 0) \circ (1 \circ 2) = \{2\} \circ \{1\} = \{2\}\). By hypothesis \(\{2\} < \{0, 1\}\). Therefore \(2 < 1\), which is a contradiction.

(c) If \( y = 2 \), then by (6), \(2 \circ 2 \subseteq \{0, 1\}\). So \((2 \circ 0) \circ (2 \circ 2) = 2 \circ (2 \circ 2) \subseteq 2 \circ \{0, 1\} = (2 \circ 0) \cup \{2\} = \{2\} \cup \{2\} = \{2\}\). By hypothesis \(\{2\} < \{0, 1\}\), hence \(2 < 1\), which is a contradiction.

(ii) Let \( z = 1 \). Then a similar argument as the case of (i), gives a contradiction.

Note that by hypothesis \(z \in I\) so \(z \neq 2\). Hence \(x = 2\) is impossible i.e., \(x \neq 2\). Thus \(x \in I\), which implies that \(I\) is an implicative hyper \(K\)-ideal. Conversely, let \(I\) be an implicative hyper \(K\)-ideal. Without loss of generality assume that \(I = \{0, 1\}\). Let \((x \circ y) \circ z < I\) and \(y \circ z < I\) for \(x, y, z \in H\), we must show that \(x \circ z \subseteq I\). By Theorem 3.17 [11], we know that \(1 \circ 0 = \{1\}, 2 \circ 0 = \{2\}, 1 \circ 2 \neq \{2\}\) and \(2 \circ 1 \neq \{1\}\). Now we show that

(I) \(1 \circ 2 = \{1\}\)

(II) \(2 \circ 1 = \{2\}\)

(III) \(x \circ y \neq \{0, 2\}, x \circ y \neq \{0, 1, 2\};\) for all \(x, y \in H\).

(I): Let \(1 \circ 2 \neq \{1\}\). Then \(1 \neq 2\), since \(H\) is simple. Thus \(0 \notin 1 \circ 2\), therefore we must have \(1 \circ 2 = \{1, 2\}\). But

\[
0 \in 2 \circ 2 \subseteq (2 \circ 1) \cup (2 \circ 2) = 2 \circ \{1, 2\} = 2 \circ (1 \circ 2) = (2 \circ 0) \circ (1 \circ 2).
\]

So \((2 \circ 0) \circ (1 \circ 2) < I\). Since \(0 \in I\), we conclude that \(2 \in I\), which is a contradiction. Hence \(1 \circ 2 = \{1\}\).

(II): Suppose \(2 \circ 1 \neq \{2\}\). Since \(2 \neq 1\), \(0 \notin 2 \circ 1\) and since \(2 \circ 1 \neq \{1\}\), thus we must have \(2 \circ 1 = \{2, 1\}\). Now \(\{1, 2\} = 2 \circ 1 = (2 \circ 0) \circ (1 \circ 2),\) by (I), that is \((2 \circ 0) \circ (1 \circ 2) < I\). Since \(0 \in I\) and \(I\) is implicative we get that \(2 \in I\) which is a contradiction. Hence \(2 \circ 1 = \{2\}\).

(III): By considering (I) and (II), it remains to show that none of \(0 \circ 0, 0 \circ 1, 1 \circ 1\) and \(2 \circ 2\) are equal to \(\{0, 2\}\) or \(\{0, 1, 2\}\). Clearly all of them contain 0, so we show that none of them contain 2.

(a) \(2 \notin 2 \circ 2\): Let \(2 \in 2 \circ 2\). Then by (II) we have \(0 \in 2 \circ 2 \subseteq 2 \circ (2 \circ 2) = (2 \circ 1) \circ (2 \circ 2),\) hence \((2 \circ 1) \circ (2 \circ 2) < I\). Since \(1 \in I\), then \(2 \in I\),
which is a contradiction. Therefore $2 \not\in 2 \circ 2$.

(b) The proof of $2 \not\in 0 \circ 2$ is similar as (a).

(c) $2 \not\in 0 \circ 1$: Let $2 \in 0 \circ 1$. Then by (HK3) and (HK2) we have $2 \in 0 \circ 1 \subseteq (2 \circ 2) \circ 1 = (2 \circ 1) \circ 2$. By (I), $(2 \circ 1) \circ 2 = 2 \circ 2$, so $2 \not\in 2 \circ 2$, which is in contradiction with (a).

(d) $2 \not\in 1 \circ 1$: Let $2 \in 1 \circ 1$. Then by (HK2) and (I) we have

\begin{equation}
2 \in 1 \circ 1 = (1 \circ 2) \circ 1 = (1 \circ 1) \circ 2.
\end{equation}

Since $0 \in 1 \circ 1$ and $2 \in 1 \circ 1$, then $1 \circ 1$ contains \{0, 2\}. Thus $1 \circ 1 = \{0, 2\}$ or \{0, 1, 2\}. If $1 \circ 1 = \{0, 1, 2\}$, then by (7), (I) and (II) we have

$2 \in (1 \circ 1) \circ 2 = \{0, 1, 2\} \circ 2 = (0 \circ 2) \cup (1 \circ 2) \cup (2 \circ 2) \subseteq \{0, 1\}$,

which is a contradiction. If $1 \circ 1 = \{0, 2\}$, then similarly we get a contradiction.

(e) $2 \not\in 0 \circ 0$: Let $2 \in 0 \circ 0$. Then by (HK2), (HK3) and (d) we have

$2 \in 0 \circ 0 \subseteq (1 \circ 1) \circ 0 = (1 \circ 0) \circ 1 = 1 \circ 1 \subseteq \{0, 1\}$, which is a contradiction. Thus (III) is proved.

Now, (III) imposes that $(H, \circ, 0)$ must have the following hyper structure table:

<table>
<thead>
<tr>
<th>\circ</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>{0} or {0,1}</td>
<td>{0} or {0,1}</td>
<td>{0} or {0,1}</td>
</tr>
<tr>
<td>1</td>
<td>{1}</td>
<td>{0} or {0,1}</td>
<td>{1}</td>
</tr>
<tr>
<td>2</td>
<td>{2}</td>
<td>{2}</td>
<td>{0} or {0,1}</td>
</tr>
</tbody>
</table>

As we see, in the above table except the cases $2 \circ 0 = \{2\}$ and $2 \circ 1 = \{2\}$, the other possible cases of $x \circ z$ are subsets of $I$. That is $x \circ z \subseteq I$. Now we prove that if $x = 2$, $z = 0$ or $x = 2$, $z = 1$, then $(x \circ y) \circ z \not\in I$, or $y \circ z \not\in I$. Therefore the proof will be completed.

First let $x = 2$ and $z = 0$. If $y = 0$, then we have

$2 = 2 \circ 0 = (2 \circ 0) \circ 0 < I = \{0, 1\}$,

which is a contradiction. Similarly for $y = 1$ or $y = 2$ we obtain a contradiction.

Now, if $x = 2$ and $z = 1$, then by a similar argument as above we give a contradiction. Hence we proved that if $(x \circ y) \circ z < I$, and $y \circ z < I$, then $x \circ z \subseteq I$, for all $x, y, z \in H$. Thus $I$ is a positive implicative hyper $K$-ideal of type 3. \qed
Corollary 4.16. Let \( H = \{0, 1, 2\} \) be a hyper \( K \)-algebra of order 3, that satisfies the simple condition and \( I \) be an implicative hyper \( K \)-ideal of \( H \). Then \( I \) is a positive hyper \( K \)-ideal of types 1-8.

Proof. The proof follows from Theorem 4.15 and Theorem 3.5 of [1].

Theorem 4.17. There are 12 non-isomorphic hyper \( K \)-algebras of order 3, with simple condition such that they have at least one proper implicative hyper \( K \)-ideal.


Theorem 4.18. Let \( I \) be an implicative hyper \( K \)-ideal of \( H \), that satisfies the strong transitive condition, \( A \) be a hyper \( K \)-ideal of \( H \) that contains \( I \). Then \( A \) is an implicative hyper \( K \)-ideal of \( H \).

Proof. Let \( x \circ (y \circ x) < A \), we prove that \( x \in A \). By Theorem 2.3 (ix) we have \( x \circ A < y \circ x \). Since \( I \subseteq A \), we get that \( x \circ I < x \circ A \), hence \( x \circ I < y \circ x \). Thus \( x \circ (y \circ x) < I \), by Theorem 2.3 (ix). Since \( I \) is an implicative hyper \( K \)-ideal we get that \( x \in I \), so \( x \in A \). Therefore by Theorem 4.12 \( A \) is an implicative hyper \( K \)-ideal of \( H \).

Theorem 4.19. If \( \{I_i| i \in \Lambda\} \) is a family of (weak) implicative hyper \( K \)-ideals, then \( \bigcap_{i \in \Lambda} I_i \) is also a (weak) implicative hyper \( K \)-ideal.

Proof. The proof is straightforward.

Theorem 4.20. Let \((H, \ast, 0)\) be a \( BCK \)-algebra and \( I \) be a nonempty subset of \( H \) which satisfies the additive condition. If we consider the hyperoperation \( x \circ y = \{x \ast y\} \) on \( H \), then \( I \) is a weak implicative hyper \( K \)-ideal of \( H \) if and only if \( I \) is an implicative hyper \( K \)-ideal of \( H \).

Proof. The proof is easy.

Open Problem. Under what suitable condition each weak implicative hyper \( K \)-ideal is an implicative hyper \( K \)-ideal?

References


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