ON SET-VALUED CHOQUET INTEGRALS
AND CONVERGENCE THEOREMS (II)

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Abstract. In this paper, we consider Choquet integrals of interval number-valued functions (simply, interval number-valued Choquet integrals). Then, we prove a convergence theorem for interval number-valued Choquet integrals with respect to an autocontinuous fuzzy measure.

1. Introduction

In this paper, we consider autocontinuity fuzzy measures [12, 15] and interval number-valued functions [16]. It is well-known that closed set-valued functions had been used repeatedly in many papers [1, 2, 5, 6, 7, 8, 9, 13, 15, 16]. Jang et al. [7, 9] studied closed set-valued Choquet integrals and convergence theorems under some sufficient conditions, for examples: (i) convergence theorems for monotone convergent sequences of Choquet integrably bounded closed set-valued functions (see [7]), (ii) convergence theorems for the upper limit and the lower limit of a sequence of Choquet integrably bounded closed set-valued functions (see [9]).

The aim of this paper is to prove a convergence theorem for convergent sequences of Choquet integrably bounded interval number-valued functions in the metric $\Delta_S$ (see Definition 3.4). In Section 2, we list various definitions and notations which are used in the proof of the convergence theorem and discuss some properties of measurable interval number-valued functions. In Section 3, using these definitions and properties, we prove the convexity of interval number-valued Choquet
integrals and discuss the concepts of convergence sequences of measurable interval number-valued functions in the metric $\triangle_S$.

2. Definitions and preliminaries

**DEFINITION 2.1.** [8, 12] (1) A fuzzy measure on a measurable space $(X, \mathcal{A})$ is an extended real-valued function $\mu : \mathcal{A} \rightarrow [0, \infty]$ satisfying

(i) $\mu(\emptyset) = 0$

(ii) $\mu(A) \leq \mu(B)$, whenever $A, B \in \mathcal{A}$, $A \subset B$.

(2) A fuzzy measure $\mu$ is said to be autocontinuous from above [resp., below] if $\mu(A \cup B_n) \rightarrow \mu(A)$ [resp., $\mu(A \sim B_n) \rightarrow \mu(A)$] whenever $A \in \mathcal{A}$, $\{B_n\} \subset \mathcal{A}$ and $\mu(B_n) \rightarrow 0$.

(3) If $\mu$ is autocontinuous both from above and from below, it is said to be autocontinuous.

Recall that a function $f : X \rightarrow [0, \infty]$ is said to be measurable if $\{x | f(x) > \alpha\} \in \mathcal{A}$ for all $\alpha \in (-\infty, \infty)$.

**DEFINITION 2.2.** [12] (1) A sequence $\{f_n\}$ of measurable functions is said to converge to $f$ in measure, in symbols $f_n \rightarrow_M f$ if for every $\epsilon > 0$, $\lim_{n \rightarrow \infty} \mu(\{x | |f_n(x) - f(x)| > \epsilon\}) = 0$.

(2) A sequence $\{f_n\}$ of measurable functions is said to converge to $f$ in distribution, in symbols $f_n \rightarrow_D f$ if for every $\epsilon > 0$, $\lim_{n \rightarrow \infty} \mu_{f_n}(r) = \mu_f(r)$ e.c., where $\mu_f(r) = \mu(\{x | f(x) > r\})$ and “e.c.” stands for “except at most countably many values of $r$”.

**DEFINITION 2.3.** [10, 11, 12] (1) The Choquet integral of a measurable function $f$ with respect to a fuzzy measure $\mu$ is defined by

$$(C) \int f \, d\mu = \int_0^\infty \mu_f(r) \, dr,$$

where the integral on the right-hand side is an ordinary one.

(2) A measurable function $f$ is called integrable if the Choquet integral of $f$ can be defined and its value is finite.

Throughout this paper, $\mathbb{R}^+$ will denote the interval $[0, \infty)$, $I(\mathbb{R}^+) = \{[a, b] | a, b \in \mathbb{R}^+ \text{ and } a \leq b\}$. Then an element in $I(\mathbb{R}^+)$ is called an interval number. On the interval number set, we define; for each pair $[a, b], [c, d] \in I(\mathbb{R}^+)$ and $k \in \mathbb{R}^+$,

$[a, b] + [c, d] = [a + c, b + d]$. 

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\[ [a, b] \cdot [c, d] = [a \cdot c, b \cdot d], \]
\[ k[a, b] = [ka, kb], \]
\[ [a, b] \leq [c, d] \text{ if and only if } a \leq c \text{ and } b \leq d. \]

Then \((I(R^+), d_H)\) is a metric space, where \(d_H\) is the Hausdorff metric defined by

\[ d_H(A, B) = \max \{ \sup_{x \in A} \inf_{y \in B} |x - y|, \sup_{y \in B} \inf_{x \in A} |x - y| \} \]

for all \(A, B \in I(R^+)\). By the definition of the Hausdorff metric, we have immediately the following proposition.

Proposition 2.4. For each pair \([a, b], [c, d] \in I(R^+)\), \(d_H([a, b], [c, d]) = \max\{|a - c|, |b - d|\}\).

Let \(C(R^+)\) be the class of closed subsets of \(R^+\). Throughout this paper, we consider a closed set-valued function \(F : X \rightarrow C(R^+)\) and an interval number-valued function \(F : X \rightarrow I(R^+)\) \(\{\emptyset\}\). We denote that \(d_H - \lim_{n \to \infty} A_n = A\) if and only if \(\lim_{n \to \infty} d_H(A_n, A) = 0\), where \(A \in I(R^+)\) and \(\{A_n\} \subset I(R^+)\).

Definition 2.5. \([1, 6, 7]\) A closed set-valued function \(F\) is said to be measurable if for each open set \(O \subset R^+\),
\[ F^{-1}(O) = \{x \in X|F(x) \cap O \neq \emptyset\} \in \mathcal{A}. \]

Definition 2.6. \([1]\) Let \(F\) be a closed set-valued function. A measurable function \(f : X \rightarrow R^+\) satisfying
\[ f(x) \in F(x) \text{ for all } x \in X \]
is called a measurable selection of \(F\).

We say \(f : X \rightarrow R^+\) is in \(L_1^c(\mu)\) if and only if \(f\) is measurable and \((C) \int_A Fd\mu < \infty\). We note that “\(x \in X \mu - a.e.\)” stands for “\(x \in X \mu\)-almost everywhere”. The property \(p(x)\) holds for \(x \in X \mu - a.e.\) means that there is a measurable set \(A\) such that \(\mu(A) = 0\) and the property \(p(x)\) holds for all \(x \in A^c\), where \(A^c\) is the complement of \(A\).

Definition 2.7. \([6, 7]\)(1) Let \(F\) be a closed set-valued function and \(A \in \mathcal{A}\). The Choquet integral of \(F\) on \(A\) is defined by
\[ (C) \int_A Fd\mu = \{(C) \int_A fd\mu| f \in S_c(F)\}, \]
where \(S_c(F)\) is the family of \(\mu - a.e.\) Choquet integrable selections of \(F\), that is,
\[ S_c(F) = \{f \in L_1^c(\mu)| f(x) \in F(x) \text{ for all } x \in X \mu - a.e.\}. \]
(2) A closed set-valued function $F$ is said to be Choquet integrable if
$$(C) \int F d\mu \neq \emptyset.$$ 

(3) A closed set-valued function $F$ is said to be Choquet integrably bounded if there is a function $g \in L^1_c(\mu)$ such that
$$\|F(x)\| = \sup_{r \in F(x)} |r| \leq g(x) \text{ for all } x \in X.$$

Instead of $(C) \int_X F d\mu$, we will write $(C) \int F d\mu$. Let us discuss some basic properties of measurable closed set-valued functions. Since $R^+ = [0, \infty)$ is a complete separable metric space in the usual topology, using Theorem 8.1.3 ([1]) and Theorem 1.0(20) ([5]), we have the following two theorems.

**Theorem 2.8.** [1, 5] A closed set-valued function $F$ is measurable if and only if there exists a sequence of measurable selections $\{f_n\}$ of $F$ such that
$$F(x) = \text{cl}\{f_n(x)\} \text{ for all } x \in X.$$

**Theorem 2.9.** [1, 5] If $F$ is a measurable closed set-valued function and Choquet integrably bounded, then it is Choquet integrable.

### 3. Main results

In this section, we prove the convexity of interval number-valued Choquet integrals and discuss the concepts of convergent sequences of measurable interval number-valued functions in the metric $\Delta_S$. Since $(X, \mathcal{A})$ is a measurable space and $R^+$ is a separable metric space, Theorem 1.0(20) ([5]) implies the following theorem. Recall that a measurable closed set-valued function is said to be convex-valued if $F(x)$ is convex for all $x \in X$ and that a set $A$ is an interval number if and only if it is closed and convex.

**Theorem 3.1.** If $F$ is a measurable closed set-valued function and Choquet integrably bounded, then there exists a sequence $\{f_n\}$ of Choquet integrable functions $f_n : X \to R^+$ such that $F(x) = \text{cl}\{f_n(x)\}$ for all $x \in X$.

**Proof.** By Theorem 1.0 (20) ([5]), there exists a sequence $\{f_n\}$ of measurable functions $f_n : X \to R^+$ such that $F(x) = \text{cl}\{f_n(x)\}$ for all...
functions are measurable, since the supremum and the infimum of a sequence $F$ that is Choquet integrably bounded and if we define $f^*(x) = \sup \{r | r \in F(x)\}$ and $f_*(x) = \inf \{r | r \in F(x)\}$ for all $x \in X$, by Proposition 3.2 ([11]),

$$(C) \int f_n d\mu \leq (C) \int g d\mu < \infty, \text{ for all } n = 1, 2, \cdots .$$

So, $f_n$ is Choquet integrable for all $n = 1, 2, \cdots$. The proof is complete.

**Theorem 3.2.** If $F$ is a measurable closed set-valued function and Choquet integrably bounded and if we define $f^*(x) = \sup \{r | r \in F(x)\}$ and $f_*(x) = \inf \{r | r \in F(x)\}$ for all $x \in X$, then $f^*$ and $f_*$ are Choquet integrable selections of $F$.

**Proof.** Since $F$ is Choquet integrably bounded, there exists a function $g \in L^1_c(\mu)$ such that $\|F(x)\| \leq g(x)$ for all $x \in X$. Theorem 3.1 implies that there is a sequence $\{f_n\}$ of Choquet integrable selections of $F$ such that $F(x) = \text{cl}\{f_n(x)\}$ for all $x \in X$. Then

$$f^*(x) = \sup \{r | r \in F(x)\} = \sup_n f_n(x)$$

and

$$f_*(x) = \inf \{r | r \in F(x)\} = \inf_n f_n(x).$$

Since the supremum and the infimum of a sequence $\{f_n\}$ of measurable functions are measurable, $f^*$ and $f_*$ are measurable. And also, we have

$$0 \leq f_*(x) \leq f^*(x) = \|F(x)\| \leq g(x) \text{ for all } x \in X.$$

Since $g \in L^1_c(\mu)$, $f^*$ and $f_*$ belong to $L^1_c(\mu)$. By the closedness of $F(x)$ for all $x \in X$, $f_*(x) \in F(x)$ and $f^*(x) \in F(x)$ for all $x \in X$. Therefore, $f^*$ and $f_*$ are Choquet integrable selections of $F$.

**Assumption (A).** For each pair $f, g \in S_c(F)$, there exists $h \in S_c(F)$ such that $f \sim h$ and $(C) \int g d\mu = (C) \int h d\mu$.

We consider the following classes of interval number-valued functions:

$\mathcal{F} = \{F | F : X \to IR^+ \text{ is measurable and Choquet integrably bounded}\}$

and

$\mathcal{F}_1 = \{F \in \mathcal{F} | F \text{ is convex - valued and satisfies the assumption(A)}\}$.

**Theorem 3.3.** If $F \in \mathcal{F}_1$, then we have

1. $cF \in \mathcal{F}_1$ for all $c \in IR^+$,
2. $(C) \int F d\mu$ is convex,
\( (C) \int F d\mu = [(C) \int f_* d\mu, (C) \int f^* d\mu] \).

**Proof.** (1) The proof of (1) is trivial.

(2) If \((C) \int F d\mu\) is a single point set, then it is convex. Otherwise, let \(y_1, y_2 \in (C) \int F d\mu\) and \(y_1 < y_2\). Then, there exist \(f_1, f_2 \in S_c(F)\) such that

\[
y_1 = (C) \int f_1 d\mu \quad \text{and} \quad y_2 = (C) \int f_2 d\mu.
\]

Further, let \(y \in (y_1, y_2)\) we need to a selection \(f \in S_c(F)\) with \(y = (C) \int f d\mu\). Since \(y \in (y_1, y_2)\), there exists \(\lambda_0 \in (0, 1)\) such that \(y = \lambda_0 y_1 + (1 - \lambda_0) y_2\). For above two selections \(f_1, f_2 \in S_c(F)\), the assumption (A) implies that there exists \(g \in S_c(F)\) such that \(f_1 \sim g\) and \((C) \int g d\mu = (C) \int f_2 d\mu\). We define a function \(f = \lambda_0 f_1 + (1 - \lambda_0) g\) and note that \(\lambda_0 f_1 \sim (1 - \lambda_0) g\). Since \(F\) is convex, \(f(x) = \lambda_0 f_1(x) + (1 - \lambda_0) g(x) \in F(x)\) for \(x \in X\) \(\mu\)-a.e. By Theorem 5.6 [11] and Proposition 3.2 (2) [11],

\[
y = \lambda_0 y_1 + (1 - \lambda_0) y_2
\]

\[
= (C) \int \lambda_0 f_1 d\mu + (C) \int (1 - \lambda_0) f_2 d\mu
\]

\[
= \lambda_0 (C) \int f_1 d\mu + (1 - \lambda_0) (C) \int f_2 d\mu
\]

\[
= \lambda_0 (C) \int f_1 d\mu + (1 - \lambda_0) (C) \int g d\mu
\]

\[
= (C) \int \lambda_0 f_1 d\mu + (C) \int (1 - \lambda_0) g d\mu
\]

\[
= (C) \int (\lambda_0 f_1 + (1 - \lambda_0) g) d\mu
\]

\[
= (C) \int f d\mu.
\]

Thus, we have \(f \in S_c(F)\) and \(y = (C) \int f d\mu \in (C) \int F d\mu\). The proof of (2) is complete.

(3) We note that \(f_* \leq f \leq f^*\) for all \(f \in S_c(F)\). Thus, by Proposition 3.2(2) [11],

\[
(C) \int f_* d\mu \leq (C) \int f d\mu \leq (C) \int f^* d\mu
\]

for all \(f \in S_c(F)\). Theorem 3.2 implies \((C) \int f_* d\mu, (C) \int f^* d\mu \in (C) \int F d\mu\). By (2), \((C) \int F d\mu\) is convex in \(R^+\) and hence \((C) \int F d\mu = [(C) \int f_* d\mu, (C) \int f^* d\mu] \).
We consider a function $\triangle_S$ on $\mathcal{F}_1$ defined by
\[
\triangle_S(F, G) = \sup_{x \in X} d_H(F(x), G(x))
\]
for all $F, G \in \mathcal{F}_1$. Then, it is easily to show that $\triangle_S$ is a metric on $\mathcal{F}_1$.

**Definition 3.4.** Let $F \in \mathcal{F}_1$. A sequence $\{F_n\} \subset \mathcal{F}_1$ converges to $F$ in the metric $\triangle_S$, in symbols, $F_n \to_{\triangle_S} F$ if
\[
\lim_{n \to \infty} \triangle_S(F_n, F) = 0.
\]

**Theorem 3.5 (Convergence Theorem).** Let $F, G, H \in \mathcal{F}_1$ and $\{F_n\}$ be a sequence in $\mathcal{F}_1$. If a fuzzy measure $\mu$ is autocontinuous and if $F_n \to_{\triangle_S} F$ and $G \leq F_n \leq H$, then we have
\[
d_H - \lim_{n \to \infty} (C) \int F_n d\mu = (C) \int F d\mu.
\]

**Proof.** By Proposition 2.4, $d_H(F_n(x), F(x)) = \max\{|f_n(x) - f(x)|, |f_n^*(x) - f^*(x)|\}$ for all $x \in X$, where $f_n(x) = \inf\{r | r \in F_n(x)\}$, $f_n^*(x) = \sup\{r | r \in F_n(x)\}$ for $n = 1, 2, \cdots$, $f_s(x) = \inf\{r | r \in F(x)\}$, and $f^*(x) = \sup\{r | r \in F(x)\}$. Since $\triangle_S(F_n, F) \to 0$ as $n \to \infty$, $\sup_{x \in X} |f_n(x) - f_s(x)| \to 0$ and $\sup_{x \in X} |f_n^*(x) - f^*(x)| \to 0$. Given any $\varepsilon > 0$, there exist two natural numbers $N_1, N_2$ such that $|f_n(x) - f_s(x)| < \varepsilon$ for all $n \geq N_1$ and all $x \in X$, and $|f_n^*(x) - f^*(x)| < \varepsilon$ for all $n \geq N_2$ and all $x \in X$. We put $N = \max\{N_1, N_2\}$. Thus for each $n \geq N$,
\[
\mu\{x | |f_n(x) - f_s(x)| \geq \varepsilon\} = \mu(\emptyset) = 0
\]
and
\[
\mu\{x | |f_n^*(x) - f^*(x)| \geq \varepsilon\} = \mu(\emptyset) = 0.
\]
Then, clearly we have that for arbitrary $\varepsilon > 0$, $\mu\{x | |f_n(x) - f_s(x)| \geq \varepsilon\} \to 0$ as $n \to \infty$ and $\mu\{x | |f_n^*(x) - f^*(x)| \geq \varepsilon\} \to 0$ as $n \to \infty$. That is, $f_n \to_{M} f_s$ and $f_n^* \to_{M} f^*$ as $n \to \infty$. It is clearly to show that if $G \leq F_n \leq H$ then $\mu_g(r) \leq f_n(r) \leq \mu_h(r)$ and $\mu_g^*(r) \leq f_n^*(r) \leq \mu_h^*(r)$ for all $r \in R^+$, where $g_s(x) = \inf\{r | r \in G(x)\}$, $g^*(x) = \sup\{r | r \in G(x)\}$, $h_s(x) = \inf\{r | r \in H(x)\}$, and $h^*(x) = \sup\{r | r \in H(x)\}$. Since $\mu$ is autocontinuous, by Theorem 3.2 [12], we have
\[
\lim_{n \to \infty} (C) \int f_n d\mu = (C) \int f_s d\mu \quad \text{and} \quad \lim_{n \to \infty} (C) \int f_n^* d\mu = (C) \int f^* d\mu.
\]
Therefore,
\[
d_H[(C) \int F_n d\mu, (C) \int F d\mu] = \max\{|(C) \int f_n d\mu - (C) \int f_s d\mu|,
\]
\[
(C) \int f_n^* d\mu - (C) \int f^* d\mu|,
\]
\[
(C) \int f_n d\mu - (C) \int f_s d\mu|,
\]
\[
(C) \int f_n^* d\mu - (C) \int f^* d\mu|,
\]
\[
(C) \int f_n d\mu - (C) \int f_s d\mu|,
\]
\[
(C) \int f_n^* d\mu - (C) \int f^* d\mu|.
\]
\[
\left| \int_{(C)} f_n^* d\mu - \int_{(C)} f^* d\mu \right| \rightarrow 0 \\
as \ n \rightarrow \infty.
\]

References


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