DEFORMATION SPACES OF 3-DIMENSIONAL FLAT MANIFOLDS

EUN SOOK KANG† AND JU YOUNG KIM

Abstract. The deformation spaces of the six orientable 3-dimensional flat Riemannian manifolds are studied. It is proved that the Teichmüller spaces are homeomorphic to the Euclidean spaces. To state more precisely, let \( \Phi \) denote the holonomy group of the manifold. Then the Teichmüller space is homeomorphic to (1) \( \mathbb{R}^6 \) if \( \Phi \) is trivial, (2) \( \mathbb{R}^4 \) if \( \Phi \) is cyclic with order two, (3) \( \mathbb{R}^2 \) if \( \Phi \) is cyclic of order 3, 4 or 6, and (4) \( \mathbb{R}^3 \) if \( \Phi \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \).

1. Preliminaries

Let \( \text{Isom}(\mathbb{R}^n) \) denote the group of isometries of the Euclidean space \( \mathbb{R}^n \). So,

\[
\text{Isom}(\mathbb{R}^n) = \mathbb{R}^n \rtimes O(n),
\]

where \( O(n) \) is the n-dimensional orthogonal group. \( \text{Isom}(\mathbb{R}^n) \) is a subgroup of the affine group

\[
\mathcal{A}(n) = \mathbb{R}^n \rtimes \text{GL}(n,\mathbb{R}).
\]

A subgroup \( \Pi \) of \( \text{Isom}(\mathbb{R}^n) \) is said to be a crystallographic group if \( \Pi \) is cocompact and discrete. A torsion free crystallographic group is called a Bieberbach group. If \( \Pi \) is a Bieberbach group of dimension \( n \), then the quotient space \( \mathbb{R}^n/\Pi \) is a Riemannian manifold of sectional curvature \( \kappa = 0 \). Conversely, a flat closed Riemannian manifold of dimension \( n \) is necessarily a quotient space of \( \mathbb{R}^n \) by a Bieberbach group of dimension \( n \), see [4].

In this paper, we focus on the 3-dimensional manifolds which are closed and Riemannian flat. We use the notation \( \mathcal{I} \) for the isometry

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group $\text{Isom}(\mathbb{R}^3)$. So,

$$I = \text{Isom}(\mathbb{R}^3) = \mathbb{R}^3 \rtimes O(3).$$

The following Bieberbach’s second theorem is crucial for us. See [3] or [4].

**Theorem 1.1 (Bieberbach).** *Any isomorphism between two crystallographic groups is a conjugation by an element of the affine group.*

There are only 10 affine diffeomorphism classes of connected closed 3-dimensional flat manifolds, six of which are orientable and the others are not. A Bieberbach group $\Pi$ contains a unique maximal normal abelian subgroup $\mathbb{Z}^3$, fitting the following commutative diagram of groups with exact rows

$$
\begin{array}{cccccc}
0 & \longrightarrow & \mathbb{Z}^3 & \longrightarrow & \Pi & \longrightarrow & \Phi & \longrightarrow & 1 \\
& & \downarrow \theta & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \rtimes O(3) & \longrightarrow & O(3) & \longrightarrow & 1,
\end{array}
$$

where $\Phi$ is called the *holonomy group* of $\Pi$. It is a finite group and $\Phi \rightarrow O(3)$ is injective.

We shall investigate the various deformation spaces associated with closed 3-dimensional flat Riemannian manifolds. Firstly, the space of discrete representations, the *Weil space*, is defined as follows:

$$\mathcal{R}(\Pi; I) = \text{the space of all injective homomorphisms } \theta \text{ of } \Pi \text{ into } I \text{ such that } \theta(\Pi) \text{ is discrete in } I \text{ and } I/\theta(\Pi) \text{ is compact.}$$

If $\Pi$ is a Bieberbach group, every element of $\mathcal{R}(\Pi; I)$ gives rise to a flat Riemannian manifold. Let $\theta, \theta' \in \mathcal{R}(\Pi; I)$. If an affine map $f : \mathbb{R}^3 \to \mathbb{R}^3$ conjugates $\theta(\Pi)$ into $\theta'(\Pi)$, then it induces an affine diffeomorphism from $\mathbb{R}^3/\theta(\Pi)$ to $\mathbb{R}^3/\theta'(\Pi)$. For $g \in I$, $\mu(g)$ denotes the conjugation by $g$. We denote the group of inner automorphisms of $I$ by $\text{Inn}(I)$. This group acts on the space $\mathcal{R}(\Pi; I)$ from the left by

$$\text{Inn}(I) \times \mathcal{R}(\Pi; I) \to \mathcal{R}(\Pi; I)$$

$$(\mu(g), \theta) \longmapsto \mu(g) \circ \theta.$$

The orbit space of this action is called the *Teichmüller space*. That is,

$$\mathcal{T}(\Pi; I) = \text{Inn}(I) \setminus \mathcal{R}(\Pi; I).$$

If $\theta, \theta' \in \mathcal{R}(\Pi; I)$ represent the same point in $\mathcal{T}(\Pi; I)$, then $\theta' = \mu(g) \circ \theta$ for some $g \in I$. This implies

$$g \circ \theta(\alpha) = \theta'(\alpha) \circ g.$$
for all $\alpha \in \mathcal{H}$. Then, $g$ induces a map

$$\theta(\mathcal{H}) \setminus \mathbb{R}^3 \rightarrow \theta'(\mathcal{H}) \setminus \mathbb{R}^3$$

which is an isometry.

2. Spaces of discrete representation

The next theorem says that there are only six 3-dimensional orientable flat manifolds.

**Theorem 2.1** ([4]). *There are just 6 affine diffeomorphism classes of compact connected orientable flat 3-dimensional Riemannian manifolds. They are represented by the manifolds $\mathbb{R}^3/\mathcal{H}$ where $\mathcal{H}$ is one of the 6 groups given below. Here $t_1$, $t_2$ and $t_3$ are translations by $a_1$, $a_2$ and $a_3$ respectively and $\Phi = \mathbb{R}^3/\mathbb{Z}^3$ is the holonomy.\**

(1) $\Phi = \{1\}$. $\mathcal{H}$ is generated by the translations $\{t_1, t_2, t_3\}$ with $\{a_i\}$ linearly independent.

(2) $\Phi = \mathbb{Z}_2$. $\mathcal{H}$ is generated by $\{t_1, t_2, t_3, \alpha\}$ where $\alpha^2 = t_1, \alpha t_2 \alpha^{-1} = t_2^{-1}$ and $\alpha t_3 \alpha^{-1} = t_3^{-1}$; $a_1$ is orthogonal to $a_2$ and $a_3$ while $\alpha = (t_{a_1/2}, A)$ with $A(a_1) = a_1, A(a_2) = -a_2, A(a_3) = -a_3$.

(3) $\Phi = \mathbb{Z}_3$. $\mathcal{H}$ is generated by $\{t_1, t_2, t_3, \alpha\}$ where $\alpha^3 = t_1, \alpha t_2 \alpha^{-1} = t_3$ and $\alpha t_3 \alpha^{-1} = t_2^{-1} t_3^{-1}$; $a_1$ is orthogonal to $a_2$ and $a_3$, $\|a_2\| = \|a_3\|$ and $\{a_2, a_3\}$ is a hexagonal plane lattice, and $\alpha = (t_{a_1/3}, A)$ with $A(a_1) = a_1, A(a_2) = a_3$ and $A(a_3) = -a_2 - a_3$.

(4) $\Phi = \mathbb{Z}_4$. $\mathcal{H}$ is generated by $\{t_1, t_2, t_3, \alpha\}$ where $\alpha^4 = t_1, \alpha t_2 \alpha^{-1} = t_3$ and $\alpha t_3 \alpha^{-1} = t_2^{-1}$; $\{a_i\}$ are mutually orthogonal with $\|a_2\| = \|a_3\|$ while $\alpha = (t_{a_1/4}, A)$ with $A(a_1) = a_1, A(a_2) = a_3$ and $A(a_3) = -a_2$.

(5) $\Phi = \mathbb{Z}_6$. $\mathcal{H}$ is generated by $\{t_1, t_2, t_3, \alpha\}$ where $\alpha^6 = t_1, \alpha t_2 \alpha^{-1} = t_3$ and $\alpha t_3 \alpha^{-1} = t_2^{-1} t_3$; $a_1$ is orthogonal to $a_2$ and $a_3$, $\|a_2\| = \|a_3\|$ and $\{a_2, a_3\}$ is a hexagonal plane lattice, and $\alpha = (t_{a_1/6}, A)$ with $A(a_1) = a_1, A(a_2) = a_3$ and $A(a_3) = a_3 - a_2$.

(6) $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathcal{H}$ is generated by $\{t_1, t_2, t_3, \alpha_1, \alpha_2, \alpha_3\}$ where $\alpha_3 \alpha_2 \alpha_1 = t_1 t_3$ and for $i = 1, 2, 3$

$$\alpha_i^2 = t_i \text{ and } \alpha_i t_j \alpha_i^{-1} = t_j^{-1} (i \neq j).$$

The $\{a_i\}$ are mutually orthogonal and

$\alpha_1 = (A_1, t_{a_1/2})$ with $A_1(a_1) = a_1, A_1(a_2) = -a_2, A_1(a_3) = -a_3$;

$\alpha_2 = (A_2, t_{(a_2+a_3)/2})$ with $A_2(a_1) = -a_1, A_2(a_2) = a_2, A_2(a_3) = -a_3$;

$\alpha_3 = (A_3, t_{(a_1+a_2+a_3)/2})$ with $A_3(a_1) = -a_1, A_3(a_2) = -a_2, A_3(a_3) = a_3$. 
We need some notation. Let $X$ be a $3 \times 3$ matrix whose column vectors are the $x_i (i = 1, 2, 3)$. Then $X^T X$ is symmetric of which the $(i, j)$ entry is the inner product $\langle x_i, x_j \rangle$ of two vectors $x_i$ and $x_j$. For subgroups $H_1, H_2$ of $G$, we denote

$$H_1 \cdot H_2 = \{ h_1 \cdot h_2 \mid h_1 \in H_1, h_2 \in H_2 \}.$$  

Note that $H_1 \cdot H_2$ is not a subgroup but a subspace of $G$. Of course $H_1$ and $H_2$ may have a nontrivial subgroup in common.

Consider a $3 \times 3$ non-singular matrix $A$. Let $X(A)$ be the space consisting of $3 \times 3$ invertible matrices by which the conjugates of $A$ are orthogonal. That is

$$X(A) = \{ X \in \text{GL}(3, \mathbb{R}) \mid XAX^{-1} \in \text{O}(3) \} = \{ X \in \text{GL}(3, \mathbb{R}) \mid (X^T X)A = A(X^T X) \}.$$  

It is easy to see that $X(P^{-1}AP) = X(A) \cdot P$.

**Theorem 2.2.** Let $\Pi \subset \mathbb{R}^3 \rtimes \text{SO}(3)$ be a 3-dimensional Bieberbach group with holonomy group $\Phi \subset \text{SO}(3)$. Let

1. $X(\Phi) = \{ X \in \text{GL}(3, \mathbb{R}) \mid XAX^{-1} \in \text{O}(3) \text{ for all } A \in \Phi \}$, and
2. $(\mathbb{R}^3)^\Phi$ be the fixed point set of the $\Phi$ action on $\mathbb{R}^3$.

Then $R(\Pi; I) = \mathbb{R}^3 \rtimes X(\Phi)/(\mathbb{R}^3)^\Phi$.

**Proof.** Let $\theta_0 : \Pi \hookrightarrow I$ be the embedding given in Theorem 2.1, and let $\theta \in R(\Pi; I)$. By Theorem 1.1, they are conjugate by an affine motion. That is, there exists an element $\xi = (x, X) \in \text{Aff}(3) = \mathbb{R}^3 \rtimes \text{GL}(3, \mathbb{R})$ such that $\theta(\Pi) = \xi \cdot \theta_0(\Pi) \cdot \xi^{-1}$. So we need to find all elements $\xi \in \text{Aff}(3)$ which conjugates $\theta_0(\Pi)$ into $I$.

Note that the fact $\xi \cdot \theta_0(\Pi) \cdot \xi^{-1} \subset I$ depends only on the matrix part of $\xi$. That is $\xi \cdot \theta_0(\Pi) \cdot \xi^{-1} \subset I$ if and only if $X\Phi X^{-1} \subset \text{O}(3)$, or equivalently, $X \in X(\Phi)$. Observe that $X(\Phi)$ is not a subgroup in general, but is a nice algebraic sub-variety of $\text{GL}(3, \mathbb{R})$. Thus the space of all $\xi \in \text{Aff}(3)$ which conjugates $\theta_0(\Pi)$ into $I$ is

$$\mathbb{R}^3 \rtimes X(\Phi).$$

Suppose now two elements $(d_1, D_1), (d_2, D_2) \in \mathbb{R}^3 \rtimes X(\Phi)$ yield the same representation. We must have

$$(d_1, D_1)(x, X)(d_1, D_1)^{-1} = (d_2, D_2)(x, X)(d_2, D_2)^{-1}$$

for all $(x, X) \in \Pi$. Since $\Pi$ contains a lattice $\mathbb{Z}^3 \subset \mathbb{R}^3$ (in fact, $\Pi/\mathbb{Z}^3 = \Phi$), the above equality for $(x, X) = (z, I)$ for all $z \in \mathbb{Z}^3$ ensures that
\[ D_1 = D_2 (= D). \] Now let us let
\[ (d_2, D) = (d_1, D)(c, I). \]

Then
\[ (d_1, D)(x, X)(d_1, D)^{-1} = (d_2, D)(x, X)(d_2, D)^{-1} \]
\[ = ((d_1, D)(c, I))(x, X)((d_1, D)(c, I))^{-1} \]
\[ = (d_1, D)((c, I)(x, X)(c, I)^{-1}) (d_1, D)^{-1}. \]

Therefore,
\[ (c, I)(x, X)(c, I)^{-1} = (x, X) \]
for all \((x, X) \in \Pi\). This readily implies that \(Xc = c\). Thus, \(c \in (\mathbb{R}^3)^{\Phi}\), the subspace of \(\mathbb{R}^3\) which is fixed by every element of \(\Phi\). Note that this is the centralizer \(C_{\text{Aff}(\mathbb{R}^3)}(\theta_0(\Pi))\) of \(\theta_0(\Pi)\) in the group \(\text{Aff}(3)\).

We can summarize this as follows: The group \(C = (\mathbb{R}^3)^{\Phi}\) acts on \(\mathbb{R}^3 \rtimes X(\Phi)\) on the right by
\[ (\mathbb{R}^3 \rtimes X(\Phi)) \times C \to \mathbb{R}^3 \rtimes X(\Phi) \]
\[ ((x, X), (c, I)) \mapsto (x, X) \cdot (c, I) = (x + Xc, X). \]

Note that \((x, X)\) and \((x, X) \cdot (c, I)\) yield the same representation. More precisely, for any \(u \in \theta_0(\Pi)\),
\[ \mu((x, X) \cdot (c, I))(u) = \mu(x, X)(\mu(c, I)(u)) \]
\[ = \mu(x, X)(u) \]
because \((c, I)\) centralizes \(\theta_0(\Pi)\). The space of representations is thus the orbit space of the action of \((\mathbb{R}^3)^{\Phi}\) on \(\mathbb{R}^3 \rtimes X(\Phi)\); that is, \(\mathcal{R}(\Pi; I) = \mathbb{R}^3 \rtimes X(\Phi)/(\mathbb{R}^3)^{\Phi}\).

**Proposition 2.3.** Let \(\Pi \subset \mathbb{R}^3 \rtimes \text{SO}(3)\) be a 3-dimensional Bieberbach group with holonomy group \(\Phi \subset \text{SO}(3)\). Then, with respect to the representation given in Theorem 2.1,

1. If \(\Phi\) is trivial, then \(X(\Phi) = \text{GL}(3, \mathbb{R})\) and \((\mathbb{R}^3)^{\Phi} = \mathbb{R}^3\).
2. If \(\Phi = \mathbb{Z}_2\), then \(X(\Phi) = \text{O}(3) \cdot (\mathbb{R}^* \times \text{GL}(2, \mathbb{R}))\) and \((\mathbb{R}^3)^{\Phi} = \mathbb{R}^1 \cong \mathbb{R}^1\).
3. If \(\Phi = \mathbb{Z}_3\), \(\mathbb{Z}_4\), or \(\mathbb{Z}_6\), then \(X(\Phi) = \text{O}(3) \cdot (\mathbb{R}^* \times (\mathbb{R}^+ \times \text{O}(2)))\) and \((\mathbb{R}^3)^{\Phi} = \mathbb{R}^1 \cong \mathbb{R}^1\).
4. If \(\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2\), then \(X(\Phi) = \text{O}(3) \cdot (\mathbb{R}^*)^3\) and \((\mathbb{R}^3)^{\Phi} = 0\).
Proof. (1) Case of $\Phi = \{1\}$: Obvious.

(2) Case of $\Phi = \mathbb{Z}_2$: The holonomy group $\mathbb{Z}_2$ is generated by $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. We have to find the matrix $X$ such that $XAX^{-1}$ is orthogonal, which is equivalent to $(X^T X)A = A(X^T X)$. This implies that two inner products $\langle x_1, x_2 \rangle$ and $\langle x_1, x_3 \rangle$ must be zero. Hence, $X(\Phi) = \{X = [x_1, x_2, x_3] \in \text{GL}(3, \mathbb{R}) \mid x_1 \perp x_2 \text{ and } x_1 \perp x_3\} = O(3) \cdot (\mathbb{R}^* \times \text{GL}(2, \mathbb{R}))$, where $\mathbb{R}^*$ means the set of all non-zero real numbers. Note that a 3-dimensional space $O(3)$ and a 5-dimensional space $\mathbb{R}^* \times \text{GL}(2, \mathbb{R})$ intersects the common space $\mathbb{Z}_2 \times O(2)$ which is 1-dimensional, and $X(\Phi)$ has 4-components. And so $O(3) \cdot (\mathbb{R}^* \times \text{GL}(2, \mathbb{R}))$ is 7-dimensional. $(\mathbb{R}^3)^\Phi = \mathbb{R}$ consists of the first axis.

(3) Cases of $\Phi = \mathbb{Z}_3$, $\mathbb{Z}_4$ and $\mathbb{Z}_6$: Denote

$$
\rho(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}
$$

the rotation matrix of rotation of $\mathbb{R}^2$ about the origin through $\theta$. Now we can take embeddings $\theta_0 : \Pi \to \mathcal{I}$ as follows: Let $\Phi = \mathbb{Z}_n$ (for $n = 3, 4$ or 6). Then

$$
\begin{align*}
\theta_0(t_1) &= (e_1, I) \\
\theta_0(t_2) &= (Ae_2, I) \\
\theta_0(t_3) &= (A^2e_2, I) \\
\theta_0(\alpha) &= (\frac{1}{n} e_1, A),
\end{align*}
$$

where $A = \begin{bmatrix} 1 & \rho(\frac{2\pi}{n}) \\ \rho(\frac{2\pi}{n}) & 1 \end{bmatrix}$. The defining condition $X \in X(\Phi)$ is $X(\Phi)X^{-1} \in O(3)$. Clearly $(X\rho(\theta)X^{-1})^T(X\rho(\theta)X^{-1}) = I$ if and only if $\rho(-\theta)X^T X\rho(\theta) = X^T X$ i.e., $(X^T X)\rho(\theta) = \rho(\theta)(X^T X)$. The most general $X^T X$ is of the form

$$
\begin{bmatrix}
a & 0 & 0 \\
0 & b & 0 \\
0 & 0 & b
\end{bmatrix},
$$

where $a, b \in \mathbb{R}$.
where \( a \) and \( b \) are nonzero real numbers. So the nonzero column vectors of \( X \) satisfies that \( x_i \perp x_j (i \neq j) \) and \( ||x_2|| = ||x_3|| \). This implies that

\[
\mathcal{X}(\Phi) = O(3) \cdot (\mathbb{R}^* \times (\mathbb{R}^+ \times O(2))).
\]

The 3-dimensional spaces \( O(3) \) and \( \mathbb{R}^* \times (\mathbb{R}^+ \times O(2)) \) have intersection \( \mathbb{Z}_2 \times O(2) \) which is 1-dimensional, and the product has 4-components. Hence \( \mathcal{X}(\Phi) \) is 5-dimensional. As in the case of \( \Phi = \mathbb{Z}_2 \), the centralizer \((\mathbb{R}^3)^\Phi = \{(x, I)|x = [*, 0, 0]^T\} \approx \mathbb{R}e_1.\)

(4) Case of \( \Phi = \mathbb{Z}_2 \times \mathbb{Z}_2 \): The holonomy group \( \Phi = \mathbb{Z}_2 \times \mathbb{Z}_2 \) is generated by

\[
A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.
\]

We look for the space

\[
\mathcal{X}(\Phi) = \{X \in GL(3, \mathbb{R}) \mid X A_i X^{-1} \text{ is orthogonal for } i = 1, 2\}.
\]

\( X A_i X^{-1} \) is orthogonal for \( 1 \leq i \leq 2 \) if and only if \( X^T X \) is diagonal. Hence

\[
\mathcal{X}(\Phi) = \{X \in GL(3, \mathbb{R}) \mid x_i \perp x_j \text{ if } i \neq j\}
\]

\[
= O(3) \cdot (\mathbb{R}^*)^3,
\]

where \((\mathbb{R}^*)^3\) is the diagonal matrices of non-zero determinant. The 3-dimensional spaces \( O(3) \) and \((\mathbb{R}^*)^3\) have intersection \((\mathbb{Z}_2)^3\), consisting of all diagonal matrices with entries \( \pm 1 \). This space is 0-dimensional. And so \( \mathcal{X}(\Phi) \) is 6-dimensional. Clearly, the centralizer \((\mathbb{R}^3)^\Phi\) is trivial. \(\square\)

**Corollary 2.4.** Let \( M \) be a 3-dimensional orientable flat manifold with \( \pi_1(M) = \Pi \). Then

1. If \( \Phi \) is trivial, then \( R(\Pi; I) = GL(3, \mathbb{R}) \), and so it is a 9-dimensional space.
2. If \( \Phi = \mathbb{Z}_2 \), then \( R(\Pi; I) = \mathbb{R}^3 \times (O(3) \cdot (\mathbb{R}^* \times GL(2, \mathbb{R}))) / \mathbb{R} \times \{1\} \), and so it is a 9-dimensional space.
3. If \( \Phi = \mathbb{Z}_3, \mathbb{Z}_4 \) or \( \mathbb{Z}_6 \), then \( R(\Pi; I) = \mathbb{R}^3 \times (O(3) \cdot (\mathbb{R}^* \times (\mathbb{R}^+ \times O(2))) / \mathbb{R} \times \{1\} \), and so it is a 7-dimensional space.
4. If \( \Phi = \mathbb{Z}_2 \times \mathbb{Z}_2 \), then \( R(\Pi; I) = \mathbb{R}^3 \times (O(3) \cdot (\mathbb{R}^*)^3) \) and so it is a 9-dimensional space.

**Remark 2.5.** In the cases of (2) and (3) of Corollary 2.4, note that the (right) action of \( \mathbb{R} \times \{I\} \cong \mathbb{R} \) on \( R^3 \times X(\Phi) \) is twisted. In other words, one cannot write the orbit space as \((\mathbb{R}^3 \times X(\Phi)) / \mathbb{R} \times \{I\} \approx \mathbb{R}^2 \times X(\Phi).\) This is because

\[
(x, X) \cdot (c, I) = (x + Xc, X)
\]
which is different from \((x + c, X)\). However the action is free and proper so that the orbit space is a manifold.

3. Teichmüller spaces

**Theorem 3.1.** Let \(M\) be a 3-dimensional orientable flat manifold with \(\Pi(M) = \Pi\). Then the Teichmüller spaces are as follow:

1. If \(\Phi\) is trivial, then \(T(\Pi; I) = O(3) \setminus GL(3, \mathbb{R}) \approx \mathbb{R}^6\).
2. If \(\Phi = \mathbb{Z}_2\), then \(T(\Pi; I) = \mathbb{R}^+ \times (O(2) \setminus GL(2, \mathbb{R})) \approx \mathbb{R}^+ \times \mathbb{R}^3 \approx \mathbb{R}^4\).
3. If \(\Phi = \mathbb{Z}_3, \mathbb{Z}_4\) or \(\mathbb{Z}_6\), then \(T(\Pi; I) = (\mathbb{R}^+)^2 \approx \mathbb{R}^2\).
4. If \(\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2\), then \(T(\Pi; I) = (\mathbb{R}^*)^3 / (\mathbb{Z}_2)^3 = (\mathbb{R}^+)^3 \approx \mathbb{R}^3\).

**Proof.** The group of isometries \(I = \mathbb{R}^3 \rtimes O(3)\) acts on \(R(\Pi; I)\) on the left by conjugation, and the orbit space is the Teichmüller space of \(\Pi\). On the space \(\mathbb{R}^3 \times X(\Phi)\)-level, this action is just a multiplication from the left. On the other hand, we had an action of \((\mathbb{R}^3)^\Phi\) by right multiplication to get the orbit space \(R(\Pi; I)\). Clearly, these two action of \(I\) and \((\mathbb{R}^3)^\Phi\) on \(\mathbb{R}^3 \times X(\Phi)\) commute with each other. Furthermore, from \(\mathbb{R}^3 \times \{I\} \subset \mathbb{R}^3 \times O(3)\), every orbit must contain whole \(\mathbb{R}^3\). Therefore,

\[
T(\Pi; I) = I \setminus (\mathbb{R}^3 \times X(\Phi)) / (\mathbb{R}^3)^\Phi
= O(3) \setminus (\mathbb{R}^3 \times \{I\} \setminus \mathbb{R}^3 \times X(\Phi)) / (\mathbb{R}^3)^\Phi
= O(3) \setminus X(\Phi).
\]

Also recall that \(O(n)\) is a maximal compact subgroup of \(GL(n, \mathbb{R})\), and \(O(n) \setminus GL(n, \mathbb{R}) \approx \mathbb{R}^{n(n+1)/2}\).

**Remark 3.2.** Here we studied the deformation spaces of groups \(\Pi \subset I\). Since all of our groups lie in \(I_0 = \mathbb{R}^3 \rtimes SO(3)\) (the connected component of the identity element, which is the orientation-preserving isometries), one may want to understand the deformation spaces with respect to \(I_0\) (instead of \(I\)).

**Remark 3.3.** For a Bieberbach group \(\Pi\), we have worked with a specific representation \(\theta_0 : \Pi \to I\) to calculate the Teichmüller space. What will happen to these deformation spaces if we use a different representation? Let \(\theta_1 : \Pi \to I\) be another discrete cocompact representation. Then, there exists \(\sigma = (d, D) \in Aff(3)\) such that

\[
\theta_1(a, A) = (d, D)\theta_0(c, A)(d, D)^{-1}
\]
for every \((a, A)\in\Pi\). For simplicity, let us denote \(\theta_0(\Pi)\) and \(\theta_1(\Pi)\) by \(\Pi\) and \(\Pi'\) respectively. Also their holonomies are denoted by \(\Phi\) and \(\Phi'\).

Then, clearly, 
\[
\mathcal{X}(\Phi') = \mathcal{X}(D\Phi D^{-1}) = \mathcal{X}(\Phi)D^{-1}
\]

so that 
\[
\mathbb{R}^3 \times \mathcal{X}(\Phi') = \mathbb{R}^3 \times \mathcal{X}(\Phi) \cdot D^{-1} = (\mathbb{R}^3 \times \mathcal{X}(\Phi)) \cdot (d, D)^{-1}.
\]

The left actions of \(I\) do not change. The normalizers are 
\[
N(\Pi') = (d, D) \cdot N(\Pi) \cdot (d, D)^{-1}.
\]

In particular, the centralizers are 
\[
(\mathbb{R}^3)^{\Phi'} \times \{I\} = (d, D) \cdot (\mathbb{R}^3)^{\Phi'} \times \{I\} \cdot (d, D)^{-1} = D((\mathbb{R}^3)^{\Phi}) \times \{I\}.
\]

The diagram 
\[
\begin{array}{ccc}
(\mathbb{R}^3 \times \mathcal{X}(\Phi), N(\Phi)) & \longrightarrow & (\mathbb{R}^3 \times \mathcal{X}(\Phi)) \\
\downarrow & & \downarrow \\
(\mathbb{R}^3 \times \mathcal{X}(\Phi'), N(\Phi')) & \longrightarrow & (\mathbb{R}^3 \times \mathcal{X}(\Phi'))
\end{array}
\]

which is given by 
\[
((b, B), (n, N)) \longrightarrow (b, B) \cdot (n, N) \\
\downarrow \\
((b, B)\sigma^{-1}, \sigma(n, N)\sigma^{-1}) \longrightarrow ((b, B) \cdot (n, N))\sigma^{-1}
\]

is commutative. This shows that the deformations spaces are precisely the right translates of the original deformation spaces by \((d, D)^{-1}\).

References


Eun Sook Kang
Korea University
Chungnam 339-800, Korea
E-mail: kes@korea.ac.kr

Ju Young Kim
Catholic University of Daegu
Daegu 712-702, Korea
E-mail: jykim@cuth.cataegu.ac.kr