NOTE ON THE RESULTS WITH LOWER SEMI-CONTINUITY

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Abstract. In this paper, we introduce the concept of lower semi-continuous from above functions and show that many well-known results, such as Ekland’s and Caristi’s theorems, remain also true under lower semi-continuous from above functions.

1. Lower semi-continuous from above functions

In what follows, let \((X, d)\) be a metric space. The lower semi-continuous condition plays a key role and has been widely used in finding the solution of \(\min_{x \in X} f(x)\). See, for example, [1]-[4] and [7]. First, we recall the definition of lower semi-continuity here.

**Definition 1.1.** A function \(f : X \rightarrow \mathbb{R}\) is said to be lower semi-continuous at \(x_0\) if, for any sequence \((x_n)\) in \(X\) with \(x_n \rightarrow x_0\),

\[
f(x_0) \leq \lim_{n \rightarrow \infty} f(x_n).
\]

Although the lower semi-continuous condition is important, it is not essential for solving some minimization problems. A function which may not be necessarily lower semi-continuous can still obtain its infimum.

The purpose of this paper is to give a generalization of lower semi-continuous functions and to show that many well-known results, such as Ekland’s and Caristi’s theorems ([5], [6]) are also true under the condition of the lower semi-continuity from above. Let us introduce the following definition:

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Definition 1.2. A function \( f : X \to \mathbb{R} \) is said to be lower semi-continuous from above at \( x_0 \) if \( x_n \to x_0 \) and \( f(x_1) \geq f(x_2) \geq \cdots \geq f(x_n) \geq \cdots \) imply that
\[
 f(x_0) \leq \lim_{n \to \infty} f(x_n).
\]

It is obvious that the lower semi-continuity implies the lower semi-continuity from above. The following example shows that the reverse is not true. Thus the lower semi-continuity from above is weaker than the lower semi-continuity.

Example 1.3. Let \( f : \mathbb{R} \to \mathbb{R} \) be defined as follows:
\[
 f(x) = \begin{cases} 
 x + \frac{1}{2} & \text{if } x < 0, \\
 x^2 + 1 & \text{if } x \geq 0.
\end{cases}
\]

It is easy to check that the function \( f \) is lower semi-continuous from above at 0, but not lower semi-continuous at 0.

Example 1.3 also shows that the epi-graph of \( f \) (shortly, \( \text{epi}(f) \)) is not closed. For definition of epi-graph, see [7]. It is well-known that the lower semi-continuity of a function is equivalent to the closedness of its epi-graph ([7]).

Proposition 1.4. Let \( D \) be a compact subset of \( X \) and a function \( f : D \to \mathbb{R} \) be lower semi-continuous from above and bounded from below. Then there exists \( x_0 \in D \) such that \( f(x_0) = \inf_{x \in D} f(x) \).

Proof. Since \( D \) is compact and \( f \) is bounded from below, there exists a sequence \( (x_n) \) in \( D \) such that \( x_n \to x_0 \in D \), \( f(x_1) \geq f(x_2) \geq \cdots \geq f(x_n) \geq \cdots \) and \( f(x_n) \to \inf_{x \in D} f(x) \). By the lower semi-continuity from above, we have
\[
 f(x_0) \leq \lim_{n \to \infty} f(x_n) = \inf_{x \in D} f(x).
\]
Hence \( f(x_0) = \inf_{x \in D} f(x) \). This completes the proof.

In normed linear spaces, we can introduce the concept of the weak lower semi-continuity from above.
Definition 1.5. Let $X$ be a normed linear space. A function $f : X \to \mathbb{R}$ is said to be \textit{weak lower semi-continuous from above} at $x_0$ if $x_n \rightharpoonup x_0$ and $f(x_1) \geq f(x_2) \geq \cdots \geq f(x_n) \geq \cdots$ imply that $f(x_0) \leq \lim_{n \to \infty} f(x_n)$, where $\rightharpoonup$ represents the weak convergence in $X$.

It is well-known that, for convex functions, the lower semi-continuity is equivalent to the weak lower semi-continuity (see [7]), but we can not prove that the lower semi-continuity from above is also equivalent to the weak lower semi-continuity from above. We conjecture that this might be true.

The following results can be viewed as generalizations of the corresponding results for lower semi-continuous convex functions:

Theorem 1.6. Let $X$ be a real reflexive Banach space and $f : D(f) \to \mathbb{R}$ be a proper lower semi-continuous from above and convex function. Suppose that $\lim_{\|x\| \to \infty} f(x) = +\infty$. Then there exists $x_0 \in D(f)$ such that $f(x_0) = \inf_{x \in D(f)} f(x)$.

Proof. Take $x_n \in D(f)$ for $n = 1, 2, \cdots$ such that

\[
\begin{align*}
    f(x_1) &\leq \inf_{x \in D(f)} f(x) + \frac{1}{2}, \\
    f(x_2) &\leq \min \left\{ f(x_1), \inf_{x \in D(f)} f(x) + \frac{1}{2^2} \right\}, \\
    f(x_3) &\leq \min \left\{ \min_{x \in \text{co}\{x_1, x_2\}} f(x), \inf_{x \in D(f)} f(x) + \frac{1}{2^3} \right\}, \\
    \cdots ,
\end{align*}
\]

\[
\begin{align*}
    f(x_{n+1}) &\leq \min \left\{ \min_{x \in \text{co}\{x_1, x_2, \cdots, x_n\}} f(x), \inf_{x \in D(f)} f(x) + \frac{1}{2^{n+1}} \right\}, \\
    \cdots .
\end{align*}
\]

By assumption, since $\lim_{\|x\| \to \infty} f(x) = +\infty$, the sequence $(x_n)$ is bounded. Since $X$ is reflexive, without loss of generality, we may assume $x_n \rightharpoonup y_0$ (otherwise, taking subsequence).
In view of \( y_0 \in \text{co}\{x_k, k \geq n\} \) for \( n = 1, 2, \cdots \), there exist a sequence \((n_k)\) of positive integers with \( n_1 < n_2 < \cdots \) and \( y_{n_k} \in \text{co}\{x_{n_1}, \cdots, x_{n_k}\}, \)
\( n_{k-1} < n_k \leq n_k, k \geq 2 \), such that \( y_{n_k} \to y_0 \). By construction of the sequence \((x_n)\), we know that \((f(y_{n_k}))\) is decreasing and so it follows from the lower semicontinuity from above of \( f \) that
\[
f(y_0) \leq \lim_{k \to \infty} f(y_{n_k}) = \inf_{x \in D(f)} f(x).
\]
This completes the proof. \( \Box \)

**Theorem 1.7.** Let \( X \) be a real normed linear space and \( f : X \to \mathbb{R} \) be a lower semi-continuous from above and convex function. Suppose that there exist \( x_0 \in D(f) \) and \( r_0 > 0 \) such that
\[
\inf_{x \in B(x_0, r_0)} f(x) > -\infty.
\]
Then there exist \( g \in X^* \) and \( b \in \mathbb{R} \) such that \( f(x) > g(x) + b \) for all \( x \in X \).

**Proof.** Since \( \inf_{x \in B(x_0, r_0)} f(x) > -\infty \), there exists \( a \in \mathbb{R} \) such that \( f(x) > a + 1 \) for all \( x \in B(x_0, r_0) \). It is easy to see that \((x_0, a) \not\in \text{epi}(f)\). Since \( \text{epi}(f) \) is closed convex, there exist \( g \in X^* \) and \( l \in \mathbb{R} \) such that
\[
g(x_0) + la < g(x) + lf(x)
\]
for all \( x \in X \). It is obvious that \( l > 0 \) and so
\[
f(x) > -\frac{1}{l} g(x) + \frac{1}{l} (g(x_0) + la)
\]
for all \( x \in X \). This completes the proof. \( \Box \)

2. Ekland’s and Caristi’s theorems

In this section, we show that the well-known results of Ekland and Caristi are also true under the condition of the lower semi-continuity from above.
Note on the results with lower semi-continuity

**Theorem 2.1** (Ekland’s Variational Principle). Let \((X, d)\) be a complete metric space, and let \(f : X \rightarrow \mathbb{R}\) be lower semi-continuous from above and bounded from below. Then, for each \(\epsilon > 0\), \(\lambda > 0\) and \(f(u_0) \leq \inf_{x \in X} f(x) + \epsilon\), there exists \(u_1 \in X\) such that

1. \(f(u_1) \leq f(u_0)\),
2. \(f(u) > f(u_1) - \frac{\epsilon}{\lambda} d(u, u_1)\) for all \(u \neq u_1\).

**Proof.** Put \(x_0 = u_0\). We construct a sequence \((x_n)\) in \(X\) inductively as follows: Assume that we have \(x_n \in X\) satisfying (1). If \(f(u) > f(x_n) - \frac{\epsilon}{\lambda} d(u, x_n)\) for all \(u \neq x_n\), then we put \(x_{n+1} = x_n\). Otherwise, we set \(S_n = \{x : f(x) \leq f(x_n) - \frac{\epsilon}{\lambda} d(x, x_n)\}\).

Take \(x_{n+1} \in S\) such that
\[
f(x_{n+1}) - \inf_{x \in S_n} f(x) \leq \frac{1}{2} [f(x_n) - \inf_{x \in S_{n-1}} f(x)].
\]

It is easy to see that \((f(x_n))\) is decreasing, and we have
\[
\frac{\epsilon}{\lambda} d(x_n, x_{n+1}) \leq f(x_n) - f(x_{n+1}).
\]

Therefore, it follows that \((x_n)\) is a Cauchy sequence and so let \(u_1 = \lim_{n \to \infty} x_n\).

Next, we show that \(u_1\) satisfies our conclusions (1) and (2). In fact, (1) is obvious. Now we prove (2). Since \((f(x_n))\) is decreasing, by the lower semi-continuity from above of \(f\), we have
\[
f(u_1) \leq \lim_{n \to \infty} f(x_n).
\]

If (2) is not true, then there exists \(x \in X\) such that
\[
f(x) \leq f(u_1) - \frac{\epsilon}{\lambda} d(u_1, x).
\]
By construction of the sequence \((x_n)\), we have \(f(u_1) \leq f(x_n) - \frac{\epsilon}{\lambda} d(u_1, x_n)\).

Therefore, it follows that
\[
f(x) \leq f(x_n) - \frac{\epsilon}{\lambda} d(u_1, x_n) - \frac{\epsilon}{\lambda} d(u_1, x) \leq f(x_n) - \frac{\epsilon}{\lambda} d(x_n, x).
\]
Thus we have \(x \in S_n\) for \(n = 1, 2, \cdots\) and hence \(f(x) \geq \inf_{y \in S_n} f(y)\), which contradicts
\[
f(x) < f(u_1) \leq \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} \inf_{x \in S_n} f(x).
\]
Therefore, the conclusion (2) is true. This completes the proof. \(\square\)
Theorem 2.2 (Caristi’s Fixed Point Theorem). Let \((X, d)\) be a complete metric space and a function \(\phi : X \to R^+\) be lower semi-continuous from above. Suppose that a mapping \(T : X \to X\) satisfies the following:

\[
d(x, Tx) \leq \phi(x) - \phi(Tx)
\]

for all \(x \in X\). Then there exists \(x_0 \in X\) such that \(Tx_0 = x_0\).

Proof. Take \(\epsilon < 1\) and \(\lambda = 1\). By Theorem 2.1, there exists \(x_0 \in X\) such that \(\phi(x_0) \leq \inf_{x \in X} \phi(x) + \epsilon\) and \(\phi(x) > \phi(x_0) - \epsilon d(x, x_0)\) for all \(x \neq x_0\).

Now, we prove \(Tx_0 = x_0\). If \(x_0\) is not a fixed point of \(T\), then we have

1. \(d(x_0, Tx_0) \leq \phi(x_0) - \phi(Tx_0)\),
2. \(\phi(Tx_0) > \phi(x_0) - \epsilon d(Tx_0, x_0)\).

Therefore, we have

\[
d(x_0, Tx_0) < \epsilon d(x_0, Tx_0),
\]

which is a contradiction. This completes the proof. \(\Box\)

The proof of Caristi’s fixed point theorem actually shows the existence of infinite fixed points of the mapping \(T\) if we know that \(\phi\) does not obtain its infimum on \(X\), which is called Caristi’s infinite fixed points theorem.

Now we state its precise form.

Theorem 2.3 (Caristi’s Infinite Fixed Points Theorem). Let \((X, d)\) be a complete metric space and a function \(\phi : X \to R^+\) be lower semi-continuous from above. Suppose that \(\phi\) does not obtain its infimum on \(X\) and a mapping \(T : X \to X\) satisfies the following:

\[
d(x, Tx) \leq \phi(x) - \phi(Tx)
\]

for all \(x \in X\). Then \(T\) has infinite fixed points in \(X\).

Proof. Suppose that \(T\) only has finite fixed points. Let \(\text{Fix}(T)\) denote the set of all fixed points of the mapping \(T\). By Theorem 2.2, \(\text{Fix}(T)\) is non-empty. Since \(\phi\) does not obtain its infimum on \(X\), we have

\[
\inf_{x \in X} \phi(x) < \min_{x \in \text{Fix}(T)} \phi(x).
\]
Taking
\[ \epsilon < \min \{ 1, \min_{x \in \text{Fix}(T)} \phi(x) - \inf_{x \in \mathcal{X}} \phi(x) \} \]
and \( \lambda = 1 \), then, by Theorem 2.1, there exists \( x_0 \in \mathcal{X} \) such that
\[ \phi(x_0) \leq \inf_{x \in \mathcal{X}} \phi(x) + \epsilon \]
and
\[ \phi(x) > \phi(x_0) - \epsilon d(x, x_0) \]
for all \( x \in \mathcal{X} \) with \( x \neq x_0 \). It is obvious that \( x_0 \notin \text{Fix}(T) \).

On the other hand, by the same argument as in Theorem 2.2, we know that \( T x_0 = x_0 \), which is a contradiction. Therefore, the mapping \( T \) has infinite fixed points in \( \mathcal{X} \). This completes the proof. \( \square \)

References


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