FUZZY $r$-REGULAR OPEN SETS AND FUZZY ALMOST $r$-CONTINUOUS MAPS

SEOK JONG LEE AND EUN PYO LEE

ABSTRACT. We introduce the concepts of fuzzy $r$-regular open sets and fuzzy almost $r$-continuous maps in the fuzzy topology of Chattopadhyay. Also we investigate the equivalent conditions of the fuzzy almost $r$-continuity.

1. Introduction

Chang [2] introduced fuzzy topological spaces and other authors continued the investigation of such spaces. Azad [1] introduced the concepts of fuzzy regular open set and fuzzy almost $r$-continuous maps in Chang’s fuzzy topology. Chattopadhyay et al. [4] introduced another definition of fuzzy topology as a generalization of Chang’s fuzzy topology. By generalizing the definitions of Azad, we introduce the concepts of fuzzy $r$-regular open sets and fuzzy almost $r$-continuous maps in the fuzzy topology of Chattopadhyay. Then the concepts introduced by Azad become special cases of our definition. Also we investigate the equivalent conditions of the fuzzy almost $r$-continuity.

2. Preliminaries

In this paper, we denote by $I$ the unit interval $[0, 1]$ of the real line and $I_0 = (0, 1]$. A member $\mu$ of $I^X$ is called a fuzzy set in $X$. For any $\mu \in I^X$, $\mu^c$ denotes the complement $1 - \mu$. By $\tilde{0}$ and $\tilde{1}$ we denote constant maps on $X$ with value 0 and 1, respectively. All other notations are standard notations of fuzzy set theory.

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A Chang’s fuzzy topology on $X$ is a family $T$ of fuzzy sets in $X$ which satisfies the following three properties:

1. $\tilde{0}, \tilde{1} \in T$.
2. If $\mu_1, \mu_2 \in T$ then $\mu_1 \land \mu_2 \in T$.
3. If $\mu_i \in T$ for each $i$, then $\bigvee \mu_i \in T$.

The pair $(X, T)$ is called a Chang’s fuzzy topological space.

A fuzzy topology on $X$ is a map $T : I^X \rightarrow I$ which satisfies the following properties:

1. $T(\tilde{0}) = T(\tilde{1}) = 1$,
2. $T(\mu_1 \land \mu_2) \geq T(\mu_1) \land T(\mu_2)$,
3. $T(\bigvee \mu_i) \geq \bigwedge T(\mu_i)$.

The pair $(X, T)$ is called a fuzzy topological space.

For each $\alpha \in (0, 1]$, a fuzzy point $x_\alpha$ in $X$ is a fuzzy set in $X$ defined by

$$x_\alpha(y) = \begin{cases} \alpha & \text{if } y = x, \\
0 & \text{if } y \neq x. \end{cases}$$

In this case, $x$ and $\alpha$ are called the support and the value of $x_\alpha$, respectively. A fuzzy point $x_\alpha$ is said to belong to a fuzzy set $\mu$ in $X$, denoted by $x_\alpha \in \mu$, if $\alpha \leq \mu(x)$. A fuzzy point $x_\alpha$ in $X$ is said to be quasi-coincident with $\mu$, denoted by $x_\alpha q \mu$, if $\alpha + \mu(x) > 1$. A fuzzy set $\rho$ in $X$ is said to be quasi-coincident with a fuzzy set $\mu$ in $X$, denoted by $\rho q \mu$, if there is an $x \in X$ such that $\rho(x) + \mu(x) > 1$.

**Definition 2.1.** ([5]) Let $\mu$ be a fuzzy set in a fuzzy topological space $(X, T)$ and $r \in I_0$. Then $\mu$ is called

1. a fuzzy $r$-open set in $X$ if $T(\mu) \geq r$,
2. a fuzzy $r$-closed set in $X$ if $T(\mu^c) \geq r$.

**Definition 2.2.** ([3]) Let $(X, T)$ be a fuzzy topological space. For each $r \in I_0$ and for each $\mu \in I^X$, the fuzzy $r$-closure is defined by

$$\text{cl}(\mu, r) = \bigwedge \{ \rho \in I^X : \mu \leq \rho, T(\rho^c) \geq r \}.$$ 

**Definition 2.3.** ([5]) Let $(X, T)$ be a fuzzy topological space. For each $r \in I_0$ and for each $\mu \in I^X$, the fuzzy $r$-interior is defined by

$$\text{int}(\mu, r) = \bigvee \{ \rho \in I^X : \mu \geq \rho, T(\rho) \geq r \}.$$
Theorem 2.4. ([5]) For a fuzzy set $\mu$ in a fuzzy topological space $(X, T)$ and $r \in I_0$, we have:

1. $\text{int}(\mu, r)^c = \text{cl}(\mu^c, r)$.
2. $\text{cl}(\mu, r)^c = \text{int}(\mu^c, r)$.

Definition 2.5. ([5]) Let $\mu$ be a fuzzy set in a fuzzy topological space $(X, T)$ and $r \in I_0$. Then $\mu$ is said to be

1. fuzzy $r$-semiopen if there is a fuzzy $r$-open set $\rho$ in $X$ such that $\rho \leq \mu \leq \text{cl}(\rho, r)$,
2. fuzzy $r$-semiclosed if there is a fuzzy $r$-closed set $\rho$ in $X$ such that $\text{int}(\rho, r) \leq \mu \leq \rho$.

Definition 2.6. ([5]) Let $x_\alpha$ be a fuzzy point in a fuzzy topological space $(X, T)$ and $r \in I_0$. Then a fuzzy set $\mu$ in $X$ is called

1. a fuzzy $r$-neighborhood of $x_\alpha$ if there is a fuzzy $r$-open set $\rho$ in $X$ such that $x_\alpha \in \rho \leq \mu$,
2. a fuzzy $r$-quasi-neighborhood of $x_\alpha$ if there is a fuzzy $r$-open set $\rho$ in $X$ such that $x_\alpha \sqsubseteq \rho \leq \mu$.

Definition 2.7. ([5]) Let $f : (X, T) \to (Y, U)$ be a map from a fuzzy topological space $X$ to another fuzzy topological space $Y$ and $r \in I_0$. Then $f$ is called

1. a fuzzy $r$-continuous map if $f^{-1}(\mu)$ is a fuzzy $r$-open set of $X$ for each fuzzy $r$-open set $\mu$ in $Y$,
2. a fuzzy $r$-semicontinuous map if $f^{-1}(\mu)$ is a fuzzy $r$-semiopen set of $X$ for each fuzzy $r$-open set $\mu$ in $Y$,
3. a fuzzy $r$-irresolute map if $f^{-1}(\mu)$ is a fuzzy $r$-semiopen set of $X$ for each fuzzy $r$-semiopen set $\mu$ in $Y$.

All the other nonstandard definitions and notations can be found in [5] and [6].

3. Fuzzy $r$-regular open sets

We define the notions of fuzzy $r$-regular open sets and fuzzy $r$-regular closed sets, and investigate some of their properties.

Definition 3.1. Let $\mu$ be a fuzzy set in a fuzzy topological space $(X, T)$ and $r \in I_0$. Then $\mu$ is said to be

1. fuzzy $r$-regular open if $\text{int}(\text{cl}(\mu, r), r) = \mu$,
2. fuzzy $r$-regular closed if $\text{cl}(\text{int}(\mu, r), r) = \mu$. 
Theorem 3.2. Let $\mu$ be a fuzzy set in a fuzzy topological space $(X, T)$ and $r \in I_0$. Then $\mu$ is fuzzy $r$-regular open if and only if $\mu^c$ is fuzzy $r$-regular closed.

Proof. It follows from Theorem 2.4. □

Remark 3.3. Clearly, every fuzzy $r$-regular open ($r$-regular closed) set is fuzzy $r$-open ($r$-closed). That the converse need not be true is shown by the following example. The example also shows that the union (intersection) of any two fuzzy $r$-regular open ($r$-regular closed) sets need not be fuzzy $r$-regular open ($r$-regular closed).

Example 3.4. Let $X = I$ and $\mu_1, \mu_2$ and $\mu_3$ be fuzzy sets in $X$ defined by
\[
\mu_1(x) = \begin{cases} 
0 & \text{if } 0 \leq x \leq \frac{1}{2}, \\
2x - 1 & \text{if } \frac{1}{2} \leq x \leq 1;
\end{cases}
\]
\[
\mu_2(x) = \begin{cases} 
1 & \text{if } 0 \leq x \leq \frac{1}{4}, \\
-4x + 2 & \text{if } \frac{1}{4} \leq x \leq \frac{1}{2}, \\
0 & \text{if } \frac{1}{2} \leq x \leq 1;
\end{cases}
\]
and
\[
\mu_3(x) = \begin{cases} 
0 & \text{if } 0 \leq x \leq \frac{1}{4}, \\
\frac{1}{3}(4x - 1) & \text{if } \frac{1}{4} \leq x \leq 1.
\end{cases}
\]

Define $T : I^X \rightarrow I$ by
\[
T(\mu) = \begin{cases} 
1 & \text{if } \mu = 0, 1, \\
\frac{1}{2} & \text{if } \mu = \mu_1, \mu_2, \mu_1 \lor \mu_2, \\
0 & \text{otherwise}.
\end{cases}
\]

Then clearly $T$ is a fuzzy topology on $X$.

1. Clearly, $\mu_1 \lor \mu_2$ is fuzzy $\frac{1}{2}$-open. Since $\text{int}(\text{cl}(\mu_1 \lor \mu_2, \frac{1}{2}), \frac{1}{2}) = \tilde{I} \neq \mu_1 \lor \mu_2$, $\mu_1 \lor \mu_2$ is not a fuzzy $\frac{1}{2}$-regular open set.

2. Since $\text{int}(\text{cl}(\mu_1, \frac{1}{2}, \frac{1}{2})) = \text{int}(\mu_2, \frac{1}{2}) = \mu_1$ and $\text{int}(\text{cl}(\mu_2, \frac{1}{2}), \frac{1}{2}) = \text{int}(\mu_1, \frac{1}{2}, \frac{1}{2}) = \mu_2$, $\mu_1$ and $\mu_2$ are fuzzy $\frac{1}{2}$-regular open sets. But $\mu_1 \lor \mu_2$ is not a fuzzy $\frac{1}{2}$-regular open set.

3. In view of Theorem 3.2, $\mu_1^c$ and $\mu_2^c$ are fuzzy $\frac{1}{2}$-regular closed sets but $\mu_1^c \land \mu_2^c = (\mu_1 \lor \mu_2)^c$ is not a fuzzy $\frac{1}{2}$-regular closed set.
Theorem 3.5. (1) The intersection of two fuzzy $r$-regular open sets is fuzzy $r$-regular open.

(2) The union of two fuzzy $r$-regular closed sets is fuzzy $r$-regular closed.

Proof. (1) Let $\mu$ and $\rho$ be any two fuzzy $r$-regular open sets in a fuzzy topological space $X$. Then $\mu$ and $\rho$ are fuzzy $r$-open sets and hence $T(\mu \land \rho) \geq T(\mu) \land T(\rho) \geq r$. Thus $\mu \land \rho$ is a fuzzy $r$-open set. Since $\mu \land \rho \leq \text{cl}(\mu \land \rho, r)$,

$$\text{int}(\text{cl}(\mu \land \rho, r), r) \geq \text{int}(\mu \land \rho, r) = \mu \land \rho.$$ 

Now, $\mu \land \rho \leq \mu$ and $\mu \land \rho \leq \rho$ implies

$$\text{int}(\text{cl}(\mu \land \rho, r), r) \leq \text{int}(\text{cl}(\rho, r), r) = \rho.$$ 

Hence $\text{int}(\text{cl}(\mu \land \rho, r), r) \leq \mu \land \rho$. Therefore $\mu \land \rho$ is fuzzy $r$-regular open.

(2) It follows from (1) and Theorem 3.2. □

Theorem 3.6. (1) The fuzzy $r$-closure of a fuzzy $r$-open set is fuzzy $r$-regular closed.

(2) The fuzzy $r$-interior of a fuzzy $r$-closed set is fuzzy $r$-regular open.

Proof. (1) Let $\mu$ be a fuzzy $r$-open set in a fuzzy topological space $X$. Then clearly $\text{int}(\text{cl}(\mu, r), r) \leq \text{cl}(\mu, r)$ implies that

$$\text{cl}(\text{int}(\text{cl}(\mu, r), r), r) \leq \text{cl}(\mu, r) = \text{cl}(\mu, r).$$ 

Since $\mu$ is fuzzy $r$-open, $\mu = \text{int}(\mu, r)$. Also since $\mu \leq \text{cl}(\mu, r)$, $\mu = \text{int}(\mu, r) \leq \text{int}(\text{cl}(\mu, r), r)$. Thus $\text{cl}(\mu, r) \leq \text{cl}(\text{int}(\text{cl}(\mu, r), r), r)$. Hence $\text{cl}(\mu, r)$ is a fuzzy $r$-regular closed set.

(2) Similar to (1). □

Let $(X, T)$ be a fuzzy topological space. For an $r$-cut $T_r = \{\mu \in X \mid T(\mu) \geq r\}$, it is obvious that $(X, T_r)$ is a Chang’s fuzzy topological space for all $r \in \mathbb{I}_0$.

Let $(X, T)$ be a Chang’s fuzzy topological space and $r \in \mathbb{I}_0$. Recall [4] that a fuzzy topology $T^r : \mathcal{P}(X) \rightarrow \mathbb{I}$ is defined by

$$T^r(\mu) = \begin{cases} 1 & \text{if } \mu = \hat{0}, \hat{1}, \\ r & \text{if } \mu \in T - \{\hat{0}, \hat{1}\}, \\ 0 & \text{otherwise}. \end{cases}$$
Theorem 3.7. Let \( \mu \) be a fuzzy set in a fuzzy topological space \((X, T)\) and \( r \in I_0 \). Then \( \mu \) is fuzzy \( r \)-regular open (\( r \)-regular closed) in \((X, T)\) if and only if \( \mu \) is fuzzy regular open (regular closed) in \((X, T_r)\).

Proof. Straightforward. \( \square \)

Theorem 3.8. Let \( \mu \) be a fuzzy set of a Chang’s fuzzy topological space \((X, T)\) and \( r \in I_0 \). Then \( \mu \) is a fuzzy regular open (\( r \)-regular closed) in \((X, T)\) if and only if \( \mu \) is fuzzy \( r \)-regular open (\( r \)-regular closed) in \((X, T_r)\).

Proof. Straightforward. \( \square \)

4. Fuzzy almost \( r \)-continuous maps

We are going to introduce the notions of fuzzy almost \( r \)-continuous maps and investigate some of their properties. Also, we describe the relations among fuzzy almost \( r \)-continuous maps, fuzzy \( r \)-continuous maps and fuzzy \( r \)-semicontinuous maps.

Definition 4.1. Let \( f : (X, T) \to (Y, U) \) be a map from a fuzzy topological space \( X \) to another fuzzy topological space \( Y \) and \( r \in I_0 \). Then \( f \) is called

(1) a fuzzy almost \( r \)-continuous map if \( f^{-1}(\mu) \) is a fuzzy \( r \)-open set of \( X \) for each fuzzy \( r \)-regular open set \( \mu \) in \( Y \), or equivalently, \( f^{-1}(\mu) \) is a fuzzy \( r \)-closed set in \( X \) for each fuzzy \( r \)-regular closed set \( \mu \) in \( Y \),

(2) a fuzzy almost \( r \)-open map if \( f(\rho) \) is a fuzzy \( r \)-open set in \( Y \) for each fuzzy \( r \)-regular open set \( \rho \) in \( X \),

(3) a fuzzy almost \( r \)-closed map if \( f(\rho) \) is a fuzzy \( r \)-closed set in \( Y \) for each fuzzy \( r \)-regular closed set \( \rho \) in \( X \).

Theorem 4.2. Let \( f : (X, T) \to (Y, U) \) be a map and \( r \in I_0 \). Then the following statements are equivalent:

(1) \( f \) is a fuzzy almost \( r \)-continuous map.

(2) \( f^{-1}(\mu) \leq \text{int}(f^{-1}(\text{cl}(\text{int}(\mu, r)), r)) \) for each fuzzy \( r \)-open set \( \mu \) in \( Y \).

(3) \( \text{cl}(f^{-1}(\text{cl}(\text{int}(\mu, r)), r)), r) \leq f^{-1}(\mu) \) for each fuzzy \( r \)-closed set \( \mu \) in \( Y \).
Proof. (1) \(\Rightarrow\) (2) Let \(f\) be fuzzy almost \(r\)-continuous and \(\mu\) any fuzzy \(r\)-open set in \(Y\). Then
\[
\mu = \text{int}(\mu, r) \leq \text{int}(\text{cl}(\mu, r), r).
\]
By Theorem 3.6(2), \(\text{int}(\text{cl}(\mu, r), r)\) is a fuzzy \(r\)-regular open set in \(Y\). Since \(f\) is fuzzy almost \(r\)-continuous, \(f^{-1}(\text{int}(\text{cl}(\mu, r), r))\) is a fuzzy \(r\)-open set in \(X\). Hence
\[
f^{-1}(\mu) \leq f^{-1}(\text{int}(\text{cl}(\mu, r), r)) = \text{int}(f^{-1}(\text{cl}(\mu, r), r), r).
\]

(2) \(\Rightarrow\) (3) Let \(\mu\) be a fuzzy \(r\)-closed set of \(Y\). Then \(\mu^c\) is a fuzzy \(r\)-open set in \(Y\). By (2),
\[
f^{-1}(\mu^c) \leq \text{int}(f^{-1}(\text{cl}(\mu^c, r), r)), r).
\]
Hence
\[
f^{-1}(\mu) = f^{-1}(\mu^c)^c \geq \text{int}(f^{-1}(\text{cl}(\mu^c, r), r))^c = \text{cl}(f^{-1}(\text{int}(\mu, r), r), r).
\]

(3) \(\Rightarrow\) (1) Let \(\mu\) be a fuzzy \(r\)-regular closed set in \(Y\). Then \(\mu\) is a fuzzy \(r\)-closed set in \(Y\) and hence
\[
f^{-1}(\mu) \leq \text{int}(f^{-1}(\text{cl}(\mu, r), r)), r) = \text{int}(f^{-1}(\mu), r).
\]
Thus \(f^{-1}(\mu) = \text{cl}(f^{-1}(\mu), r)\) and hence \(f^{-1}(\mu)\) is a fuzzy \(r\)-closed set in \(X\). Therefore, \(f\) is a fuzzy almost \(r\)-continuous map. \(\square\)

**Theorem 4.3.** Let \(f : (X, T) \rightarrow (Y, U)\) be a map and \(r \in I_0\). Then \(f\) is fuzzy almost \(r\)-open if and only if \(f(\text{int}(\rho, r)) \leq \text{int}(f(\rho), r)\) for each fuzzy \(r\)-semiclosed set \(\rho\) in \(X\).

**Proof.** Let \(f\) be fuzzy almost \(r\)-open and \(\rho\) a fuzzy \(r\)-semiclosed set in \(X\). Then \(\text{int}(\rho, r) \leq \text{cl}(\rho, r), r) \leq \rho\). Note that \(\text{cl}(\rho, r)\) is a fuzzy \(r\)-closed set of \(X\). By Theorem 3.6(2), \(\text{int}(\text{cl}(\rho, r), r)\) is a fuzzy \(r\)-regular open set in \(X\). Since \(f\) is fuzzy almost \(r\)-open, \(f(\text{int}(\text{cl}(\rho, r), r))\) is a fuzzy \(r\)-open set in \(X\). Thus we have
\[
f(\text{int}(\rho, r)) \leq f(\text{int}(\text{cl}(\rho, r), r)) = \text{int}(f(\text{int}(\text{cl}(\rho, r), r), r) \leq \text{int}(f(\rho), r).
\]
Conversely, let $\rho$ be a fuzzy $r$-regular open set of $X$. Then $\rho$ is fuzzy $r$-open and hence $\text{int}(\rho, r) = \rho$. Since $\text{int}(\text{cl}(\rho, r), r) = \rho$, $\rho$ is fuzzy $r$-semiclosed. So

$$f(\rho) = f(\text{int}(\rho, r)) \leq \text{int}(f(\rho), r) \leq f(\rho).$$

Thus $f(\rho) = \text{int}(f(\rho), r)$ and hence $f(\rho)$ is a fuzzy $r$-open set in $Y$. \qed

The global property of fuzzy almost $r$-continuity can be rephrased to the local property in terms of neighborhood and quasi-neighborhood, respectively, in the following two theorems.

**Theorem 4.4.** Let $f : (X, T) \rightarrow (Y, U)$ be a map and $r \in I_0$. Then $f$ is fuzzy almost $r$-continuous if and only if for every fuzzy point $x_{\alpha}$ in $X$ and every fuzzy $r$-neighborhood $\mu$ of $f(x_{\alpha})$, there is a fuzzy $r$-neighborhood of $x_{\alpha}$ such that $x_{\alpha} \in \rho$ and $f(\rho) \leq \text{int}(\text{cl}(\rho, r), r)$.

**Proof.** Let $x_{\alpha}$ be a fuzzy point in $X$ and $\mu$ a fuzzy $r$-neighborhood of $f(x_{\alpha})$. Then there is a fuzzy $r$-open set $\lambda$ in $Y$ such that $f(x_{\alpha}) \in \lambda \leq \mu$. So $x_{\alpha} \in f^{-1}(\lambda) \leq f^{-1}(\mu)$. Since $f$ is fuzzy almost $r$-continuous,

$$f^{-1}(\lambda) \leq \text{int}(f^{-1}(\text{cl}(\lambda, r)), r) \leq \text{int}(f^{-1}(\text{cl}(\mu, r)), r).$$

Put $\rho = f^{-1}(\text{cl}(\mu, r), r))$. Then $x_{\alpha} \in f^{-1}(\lambda) \leq \text{int}(\rho, r) \leq \rho$. By Theorem 3.6(2), $\text{int}(\text{cl}(\mu, r), r)$ is fuzzy $r$-regular open. Since $f$ is fuzzy almost $r$-continuous, $\rho = f^{-1}(\text{cl}(\mu, r), r))$ is fuzzy $r$-open. Thus $\rho$ is a fuzzy $r$-neighborhood of $x_{\alpha}$ and

$$f(\rho) = ff^{-1}(\text{int}(\text{cl}(\mu, r), r)) \leq \text{int}(\text{cl}(\mu, r), r).$$

Conversely, let $\mu$ be a fuzzy $r$-regular open set in $Y$ and $x_{\alpha} \in f^{-1}(\mu)$. Then $\mu$ is fuzzy $r$-open and hence $\mu$ is a fuzzy $r$-neighborhood of $f(x_{\alpha})$. By hypothesis, there is a fuzzy $r$-neighborhood $\rho_{x_{\alpha}}$ of $x_{\alpha}$ such that $x_{\alpha} \in \rho_{x_{\alpha}}$ and $f(\rho_{x_{\alpha}}) \leq \text{int}(\text{cl}(\mu, r), r) = \mu$. Since $\rho_{x_{\alpha}}$ is a fuzzy $r$-neighborhood of $x_{\alpha}$, there is a fuzzy $r$-open set $\lambda_{x_{\alpha}}$ in $X$ such that

$$x_{\alpha} \in \lambda_{x_{\alpha}} \leq \rho_{x_{\alpha}} \leq f^{-1}(\rho_{x_{\alpha}}) \leq f^{-1}(\mu).$$

So we have

$$f^{-1}(\mu) = \bigvee\{x_{\alpha} : x_{\alpha} \in f^{-1}(\mu)\}$$

$$\leq \bigvee\{\lambda_{x_{\alpha}} : x_{\alpha} \in f^{-1}(\mu)\}$$

$$\leq f^{-1}(\mu).$$

Thus $f^{-1}(\mu) = \bigvee\{\lambda_{x_{\alpha}} : x_{\alpha} \in f^{-1}(\mu)\}$ is fuzzy $r$-open in $X$ and hence $f$ is almost $r$-continuous. \qed
Theorem 4.5. Let $f : (X, T) \to (Y, U)$ be a map and $r \in I_0$. Then $f$ is a fuzzy almost $r$-continuous map if and only if for every fuzzy point $x_{\alpha}$ in $X$ and every fuzzy $r$-quasi-neighborhood $\mu$ of $f(x_{\alpha})$, there is a fuzzy $r$-quasi-neighborhood $\rho$ of $x_{\alpha}$ such that $x_{\alpha}q\rho$ and $f(\rho) \leq \text{int}(\text{cl}(\mu, r), r)$.

Proof. Let $x_{\alpha}$ be a fuzzy point in $X$ and $\mu$ a fuzzy $r$-quasi-neighborhood of $f(x_{\alpha})$. Then there is a fuzzy $r$-open set $\lambda$ in $Y$ such that $f(x_{\alpha})q\lambda \leq \mu$. So $x_{\alpha}qf^{-1}(\lambda)$. Since $f$ is fuzzy almost $r$-continuous,

$$f^{-1}(\lambda) \leq \text{int}(f^{-1}(\text{int}(\text{cl}(\lambda, r))), r) \leq \text{int}(f^{-1}(\text{int}(\text{cl}(\mu, r))), r).$$

Put $\rho = f^{-1}(\text{int}(\text{cl}(\mu, r), r))$. Then $x_{\alpha}qf^{-1}(\lambda) \leq \text{int}(\rho, r) \leq \rho$. So $x_{\alpha}q\rho$. Since $\text{int}(\text{cl}(\mu, r), r)$ is fuzzy $r$-regular open and $f$ is fuzzy almost $r$-continuous, $\rho = f^{-1}(\text{int}(\text{cl}(\mu, r), r))$ is fuzzy $r$-open. Thus $\rho$ is a fuzzy $r$-quasi-neighborhood of $x_{\alpha}$ and

$$f(\rho) = ff^{-1}(\text{int}(\text{cl}(\mu, r), r)) \leq \text{int}(\text{cl}(\mu, r), r).$$

Conversely, let $\mu$ be a fuzzy $r$-regular open set in $Y$. If $f^{-1}(\mu) = \emptyset$, then it is obvious. Suppose $x_{\alpha}$ is a fuzzy point in $f^{-1}(\mu)$ such that $\alpha < f^{-1}(\mu)(x)$. Then $\alpha < \mu(f(x))$ and hence $f(x)_{1-\alpha}q\mu$. So $\mu$ is a fuzzy $r$-quasi-neighborhood of $f(x)_{1-\alpha} = f(x_{1-\alpha})$. By hypothesis, there is a fuzzy $r$-quasi-neighborhood $\rho_{x_{\alpha}}$ of $x_{1-\alpha}$ such that $x_{1-\alpha}q\rho_{x_{\alpha}}$ and $f(\rho_{x_{\alpha}}) \leq \text{int}(\text{cl}(\mu, r), r) = \mu$. Since $\rho_{x_{\alpha}}$ is a fuzzy $r$-quasi-neighborhood of $x_{1-\alpha}$, there is a fuzzy $r$-open set $\lambda_{x_{\alpha}}$ in $X$ such that

$$x_{1-\alpha}q\lambda_{x_{\alpha}} \leq \rho_{x_{\alpha}} \leq f^{-1}f(\rho_{x_{\alpha}}) \leq f^{-1}(\mu).$$

Then $\alpha < \lambda_{x_{\alpha}}(x)$ and hence $x_{\alpha} \in \lambda_{x_{\alpha}}$. So

$$f^{-1}(\mu) = \bigvee \{x_{\alpha} : x_{\alpha} \text{ is a fuzzy point in } f^{-1}(\mu) \text{ such that } \alpha < f^{-1}(\mu)(x)\}$$

$$\leq \bigvee \{\lambda_{x_{\alpha}} : x_{\alpha} \text{ is a fuzzy point in } f^{-1}(\mu) \text{ such that } \alpha < f^{-1}(\mu)(x)\}$$

$$\leq f^{-1}(\mu)$$

and hence

$$f^{-1}(\mu) = \bigvee \{\lambda_{x_{\alpha}} : x_{\alpha} \text{ is a fuzzy point in } f^{-1}(\mu) \text{ such that } \alpha < f^{-1}(\mu)(x)\}. $$

Thus $f^{-1}(\mu)$ is fuzzy $r$-open in $X$. Therefore $f$ is fuzzy almost $r$-continuous. □
Theorem 4.6. Let $f : (X, T) \rightarrow (Y, U)$ be fuzzy $r$-semicontinuous and fuzzy almost $r$-open. Then $f$ is fuzzy $r$-irresolute.

Proof. Let $\mu$ be fuzzy $r$-semiclosed in $Y$. Then $\text{int}(\text{cl}(\mu), r) \leq \mu$. Since $f$ is fuzzy $r$-semicontinuous,

$$\text{int}(\text{cl}(f^{-1}(\mu), r), r) \leq f^{-1}(\text{cl}(\mu, r)).$$

Thus we have

$$\text{int}(\text{cl}(f^{-1}(\mu), r), r) = \text{int}(\text{int}(\text{cl}(f^{-1}(\mu), r), r), r) \leq \text{int}(f^{-1}(\text{cl}(\mu, r)), r).$$

Since $f$ is fuzzy $r$-semicontinuous and $\text{cl}(\mu, r)$ is fuzzy $r$-closed, $f^{-1}(\text{cl}(\mu, r))$ is a fuzzy $r$-semiclosed set in $X$. Since $f$ is fuzzy almost $r$-open,

$$f(\text{int}(f^{-1}(\text{cl}(\mu, r)), r)) \leq \text{int}(ff^{-1}(\text{cl}(\mu, r)), r) \leq \text{int}(\text{cl}(\mu, r), r) \leq \mu.$$

Hence we have

$$\text{int}(\text{cl}(f^{-1}(\mu), r), r) \leq f^{-1}f(\text{int}(\text{cl}(f^{-1}(\mu), r), r)) \leq f^{-1}(\text{cl}(\mu, r)) \leq f^{-1}(\mu).$$

Thus $f^{-1}(\mu)$ is fuzzy $r$-semiclosed in $X$ and hence $f$ is fuzzy $r$-irresolute. □

Remark 4.7. Clearly a fuzzy $r$-continuous map is a fuzzy almost $r$-continuous map. That the converse need not be true is shown by the following example. Also, the example shows that a fuzzy almost $r$-continuous map need not be a fuzzy $r$-semicontinuous map.

Example 4.8. Let $X = I$ and $\mu_1, \mu_2$ and $\mu_3$ be fuzzy sets in $X$ defined by

$$\mu_1(x) = x;$$
$$\mu_2(x) = 1 - x;$$
and

$$\mu_3(x) = \begin{cases} x & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 0 & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$
Define $T_1 : I^X \rightarrow I$ and $T_2 : I^X \rightarrow I$ by

\[
T_1(\mu) = \begin{cases} 
1 & \text{if } \mu = \tilde{0}, \tilde{1} \\
\frac{1}{2} & \text{if } \mu = \mu_1, \mu_2, \mu_1 \lor \mu_2, \mu_1 \land \mu_2 \\
0 & \text{otherwise},
\end{cases}
\]

and

\[
T_2(\mu) = \begin{cases} 
1 & \text{if } \mu = \tilde{0}, \tilde{1} \\
\frac{1}{2} & \text{if } \mu = \mu_1, \mu_2, \mu_3, \mu_1 \lor \mu_2, \mu_1 \land \mu_2 \\
0 & \text{otherwise}.
\end{cases}
\]

Then clearly $T_1, T_2$ are fuzzy topologies on $X$. Consider the identity map $1_X : (X, T_1) \rightarrow (X, T_2)$. It is clear that $\mu_1, \mu_2, \mu_1 \lor \mu_2$ and $\mu_1 \land \mu_2$ are fuzzy $\frac{1}{2}$-regular open in $(X, T_2)$ while $\mu_3$ is not. Noting that $T_1(\mu_3) = 0$, it is obvious that $1_X$ is a fuzzy $\frac{1}{2}$-almost continuous map which is not a fuzzy $\frac{1}{2}$-continuous map. Also, because $\tilde{0}$ is the only fuzzy $\frac{1}{2}$-open set contained in $\mu_3$, $\mu_3 = 1_X^{-1}(\mu_3)$ is not a fuzzy $\frac{1}{2}$-semiopen set in $(X, T_1)$ and hence $1_X$ is not a fuzzy $\frac{1}{2}$-semicontinuous map.

**Example 4.9.** A fuzzy $r$-semicontinuous map need not be a fuzzy almost $r$-continuous map.

Let $(X, T)$ be a fuzzy topological space as described in Example 3.4 and let $f : (X, T) \rightarrow (X, T)$ be defined by $f(x) = \frac{x}{2}$. Simple computations give $f^{-1}(\tilde{0}) = \tilde{0}, f^{-1}(\tilde{1}) = \tilde{1}, f^{-1}(\mu_1) = 0$ and $f^{-1}(\mu_2) = \mu_1^\circ = f^{-1}(\mu_1 \lor \mu_2)$. Since $\text{cl}(\mu_2, \frac{1}{2}) = \mu_1^\circ, \mu_1^\circ$ is a fuzzy $\frac{1}{2}$-semiopen set and hence $f$ is a fuzzy $\frac{1}{2}$-semicontinuous map. But $f^{-1}(\mu_2) = \mu_1^\circ$ and

\[
\text{int}(f^{-1}(\text{int}(\text{cl}(\mu_2, \frac{1}{2})), \frac{1}{2})), \frac{1}{2}) = \text{int}(f^{-1}(\text{int}(\mu_1^\circ, \frac{1}{2})), \frac{1}{2})
\]

\[
= \text{int}(f^{-1}(\mu_2), \frac{1}{2})
\]

\[
= \text{int}(\mu_1^\circ, \frac{1}{2}) = \mu_2.
\]

Thus $f^{-1}(\mu_2) \not\subseteq \text{int}(f^{-1}(\text{int}(\text{cl}(\mu_2, \frac{1}{2})), \frac{1}{2})), \frac{1}{2})$ and hence $f$ is not a fuzzy almost $\frac{1}{2}$-continuous map.

From Example 4.8 and 4.9 we have the following result.

**Theorem 4.10.** Fuzzy $r$-semicontinuity and fuzzy almost $r$-continuity are independent notions.
Definition 4.11. Let \((X, T)\) be a fuzzy topological space and \(r \in I_0\). Then \((X, T)\) is called a fuzzy \(r\)-semiregular space if each fuzzy \(r\)-open set in \(X\) is a union of fuzzy \(r\)-regular open sets.

Theorem 4.12. Let \(r \in I_0\) and \(f : (X, T) \to (Y, U)\) be a map from a fuzzy topological space \(X\) to a fuzzy \(r\)-semiregular space \(Y\). Then \(f\) is fuzzy almost \(r\)-continuous if and only if \(f\) is fuzzy \(r\)-continuous.

Proof. Due to Remark 4.7, it suffices to show that if \(f\) is fuzzy almost \(r\)-continuous then it is fuzzy \(r\)-continuous. Let \(\mu\) be a fuzzy \(r\)-open set in \(Y\). Since \((Y, U)\) is a \(r\)-semiregular space, \(\mu = \bigvee \mu_i\), where \(\mu_i\)'s are fuzzy \(r\)-regular open sets in \(Y\). Then since \(f\) is a fuzzy almost \(r\)-continuous map, \(f^{-1}(\mu_i)\) is a fuzzy \(r\)-open set for each \(i\). So

\[ T(f^{-1}(\mu)) = T(f^{-1}(\bigvee \mu_i)) = T(\bigvee f^{-1}(\mu_i)) \geq \bigwedge T(f^{-1}(\mu_i)) \geq r. \]

Thus \(f^{-1}(\mu)\) is fuzzy \(r\)-open in \(X\) and hence \(f\) is a fuzzy \(r\)-continuous map. \(\square\)

Theorem 4.13. Let \(f : (X, T) \to (Y, U)\) be a map from a fuzzy topological space \(X\) to another fuzzy topological space \(Y\) and \(r \in I_0\). Then \(f\) is fuzzy almost \(r\)-continuous \((r\)-open, \(r\)-closed) if and only if \(f : (X, T_r) \to (Y, U_r)\) is fuzzy almost continuous \((open, closed)\).

Proof. Straightforward. \(\square\)

Theorem 4.14. Let \(f : (X, T) \to (Y, U)\) be a map from a Chang’s fuzzy topological space \(X\) to another Chang’s fuzzy topological space \(Y\) and \(r \in I_0\). Then \(f\) is fuzzy almost continuous \((open, closed)\) if and only if \(f : (X, T^r) \to (Y, U^r)\) is fuzzy almost \(r\)-continuous \((r\)-open, \(r\)-closed).

Proof. Straightforward. \(\square\)

References

Fuzzy $r$-regular open sets and fuzzy almost $r$-continuous maps


Seok Jong Lee, Department of Mathematics, Chungbuk National University, Cheongju 361-763, Korea
E-mail: sjlee@cbnu.ac.kr

Eun Pyo Lee, Department of Mathematics, Seonam University, Namwon 590-711, Korea
E-mail: eplee@tiger.seonam.ac.kr