SELF-ADJOINT INTERPOLATION FOR OPERATORS IN TRIDIAGONAL ALGEBRAS

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Abstract. Given operators $X$ and $Y$ acting on a Hilbert space $H$, an interpolating operator is a bounded operator $A$ such that $AX = Y$. An interpolating operator for $n$-operators satisfies the equation $AX_i = Y_i$ for $i = 1, 2, \ldots, n$. In this article, we obtained the following: Let $X = (x_{ij})$ and $Y = (y_{ij})$ be operators in $B(H)$ such that $x_{i\sigma(i)} \neq 0$ for all $i$. Then the following statements are equivalent.

1. Introduction

Let $\mathcal{C}$ be a collection of operators acting on a Hilbert space $\mathcal{H}$ and let $X$ and $Y$ be operators acting on $\mathcal{H}$. An interpolation question for $\mathcal{C}$ asks for which $X$ and $Y$ is there a bounded operator $T$ in $\mathcal{C}$ such that $TX = Y$. A variation, the ‘$n$-operator interpolation problem’, asks for an operator $T$ such that $TX_i = Y_i$ for fixed finite collections \{\$X_1, X_2, \ldots, X_n\}$ and \{\$Y_1, Y_2, \ldots, Y_n\$\}.

In this article, we investigate self-adjoint interpolation problems in tridiagonal algebras: Given operators $X$ and $Y$ acting on a Hilbert space $\mathcal{H}$, when does there exists a self-adjoint operator $A$ in a tridiagonal algebra such that $AX = Y$?
First, we establish some notations and conventions. A commutative subspace lattice $\mathcal{L}$, or CSL $L$ is a strongly closed lattice of pairwise-commuting projections acting on a Hilbert space $H$. We assume that the projections 0 and $I$ lie in $L$. We usually identify projections and their ranges, so that it makes sense to speak of an operator as leaving a projection invariant. If $\mathcal{L}$ is CSL, $\text{Alg}\mathcal{L}$ is called a CSL-algebra. The symbol $\text{Alg}\mathcal{L}$ is the algebra of all bounded linear operators on $H$ that leave invariant all the projections in $\mathcal{L}$. Let $\mathbb{N}$ be the set of all natural numbers and let $\mathbb{C}$ be the set of all complex numbers. Let $z \in \mathbb{C}$. Then $\overline{z}$ means the complex conjugate of $z$.

2. Results

Let $H$ be a separable complex Hilbert space with a fixed orthonormal basis $\{e_1, e_2, \cdots \}$. Let $x_1, x_2, \cdots, x_n$ be vectors in $H$. Then $[x_1, x_2, \cdots, x_n]$ means the closed subspace generated by the vectors $x_1, x_2, \cdots, x_n$. Let $M$ be a subset of a Hilbert space $H$. Then $\overline{M}$ means the closure of $M$ and $\overline{M}^\perp$ the orthogonal complement of $M$. Let $\mathcal{L}$ be a subspace lattice of orthogonal projections generated by the subspaces $[e_{2k-1}], [e_{2k-1}, e_{2k}, e_{2k+1}]$ ($k = 1, 2, \cdots$). Then the algebra $\text{Alg}\mathcal{L}$ is called a tridiagonal algebra which was introduced by F. Gilfeather and D. Larson [3]. These algebras have been found to be useful counterexamples to a number of plausible conjectures. Recently, such algebras have been found to be use in physics, in electrical engineering and in general system theory.

Let $A$ be the algebra consisting of all bounded operators acting on $H$ of the form

$$
\begin{pmatrix}
* & * \\
* & * & * \\
* & * \\
& & & & \\
& & \ddots
\end{pmatrix}
$$

with respect to the orthonormal basis $\{e_1, e_2, \cdots \}$, where all non-starred entries are zero. It is easy to see that $\text{Alg}\mathcal{L}=A$. Let $D=\{A: A$ is diagonal in $B(H) \}$. Then $D$ is a masa of $\text{Alg}\mathcal{L}$ and $D=(\text{Alg}\mathcal{L})\cap (\text{Alg}\mathcal{L})^*$, where

$$(\text{Alg}\mathcal{L})^* = \{A^* : A \in \text{Alg}\mathcal{L}\}.$$

In this paper, we use the convention $\frac{0}{0}=0$, when necessary.

From now, let $\sigma: \mathbb{N} \to \mathbb{N}$ be a mapping in this paper.
**Theorem 1.** Let \( X = (x_{ij}) \) and \( Y = (y_{ij}) \) be operators in \( \mathcal{B}(\mathcal{H}) \) such that \( x_{i\sigma(i)} \neq 0 \) for all \( i \). Then the following statements are equivalent.

1. There exists an operator \( A \) in \( \text{Alg}\mathcal{L} \) such that \( AX = Y \), every \( E \) in \( \mathcal{L} \) reduces \( A \) and \( A \) is a self-adjoint operator.

2. \[
\sup \left\{ \frac{\| \sum_{i=1}^{n} E_i Y f_i \|}{\| \sum_{i=1}^{n} E_i X f_i \|} : n \in \mathbb{N}, E_i \in \mathcal{L} \text{ and } f_i \in \mathcal{H} \right\} < \infty
\]
\( x_{i\sigma(i)} \) is real for all \( i \).

*Proof.* (1) \(\Rightarrow\) (2): Since \( E \) reduces \( A \) and \( AX = Y \), \( AEX = EY \) for every \( E \) in \( \mathcal{L} \). So \( A(\sum_{i=1}^{n} E_i X f_i) = \sum_{i=1}^{n} E_i Y f_i \) and hence \( \| \sum_{i=1}^{n} E_i Y f_i \| \leq \| A \| \| \sum_{i=1}^{n} E_i X f_i \| \), \( n \in \mathbb{N}, E_i \in \mathcal{L} \) and \( f_i \in \mathcal{H} \). If \( \| \sum_{i=1}^{n} E_i X f_i \| \neq 0 \), then \( \frac{\| \sum_{i=1}^{n} E_i Y f_i \|}{\| \sum_{i=1}^{n} E_i X f_i \|} \leq \| A \| \).

Hence \( \sup \left\{ \frac{\| \sum_{i=1}^{n} E_i Y f_i \|}{\| \sum_{i=1}^{n} E_i X f_i \|} : n \in \mathbb{N}, E_i \in \mathcal{L}, \text{ and } f_i \in \mathcal{H} \right\} < \infty \). Since every \( E \) in \( \mathcal{L} \) reduces \( A \), \( A \) is a diagonal operator. Let \( A = (a_{ii}) \). Since \( AX = Y \), \( y_{ij} = a_{ii} x_{ij} \) for all \( i \) and all \( j \). Since \( A \) is a self-adjoint operator, \( x_{i\sigma(i)} \) is real for all \( i \).

(2) \(\Rightarrow\) (1): If \( \sup \left\{ \frac{\| \sum_{i=1}^{n} E_i Y f_i \|}{\| \sum_{i=1}^{n} E_i X f_i \|} : n \in \mathbb{N}, E_i \in \mathcal{L} \text{ and } f_i \in \mathcal{H} \right\} < \infty \), then without loss of generality, we may assume that

\[
\sup \left\{ \frac{\| \sum_{i=1}^{n} E_i Y f_i \|}{\| \sum_{i=1}^{n} E_i X f_i \|} : n \in \mathbb{N}, E_i \in \mathcal{L} \text{ and } f_i \in \mathcal{H} \right\} = 1.
\]

So, \( \| \sum_{i=1}^{n} E_i Y f_i \| \leq \| \sum_{i=1}^{n} E_i X f_i \| \), \( n \in \mathbb{N}, E_i \in \mathcal{L} \) and \( f_i \in \mathcal{H} \) \(\cdots(*)\).

Let \( \mathcal{M} = \left\{ \sum_{i=1}^{n} E_i X f_i : n \in \mathbb{N}, E_i \in \mathcal{L} \text{ and } f_i \in \mathcal{H} \right\} \). Then \( \mathcal{M} \) is a linear manifold.

Define \( A : \mathcal{M} \rightarrow \mathcal{H} \) by \( A(\sum_{i=1}^{n} E_i X f_i) = \sum_{i=1}^{n} E_i Y f_i \). Then \( A \) is well-defined by \((*)\). Extend \( A \) to \( \overline{\mathcal{M}} \) by continuity. Define \( A|_{\overline{\mathcal{M}}} = 0 \). Then \( \| A \| \leq 1 \) and \( AX = Y \). \( AE(\sum_{i=1}^{n} E_i X f_i) = A(\sum_{i=1}^{n} E_i E_i X f_i) = \sum_{i=1}^{n} EE_i Y f_i \) and \( EA(\sum_{i=1}^{n} E_i X f_i) = E(\sum_{i=1}^{n} E_i Y f_i) = \sum_{i=1}^{n} EE_i Y f_i \).

And \( EA(g) = E(0) = 0 \) and \( AE(g) = 0 \) for \( g \) in \( \overline{\mathcal{M}} \) since \( \langle E_g, \sum_{i=1}^{n} E_i X f_i \rangle = \langle g, \sum_{i=1}^{n} E_i E_i X f_i \rangle = 0 \). Hence every \( E \) in \( \mathcal{L} \) reduces \( A \). Therefore, \( A \) is a self-adjoint operator. Let \( A = (a_{ii}) \). Since \( AX = Y \), \( y_{ij} = a_{ii} x_{ij} \) for all \( i \) and all \( j \). Since \( x_{i\sigma(i)} \) is real for all \( i \), \( A \) is a self-adjoint operator. \( \square \)
Theorem 2. Let $X_p = (x_{ij}^{(p)})$ and $Y_p = (y_{ij}^{(p)})$ be operators in $B(H)$ ($p = 1, 2, \cdots, n$) such that $x_{i\sigma(i)}^{(q)} \neq 0$ for some $q$. Then the following statements are equivalent.

1. There exists an operator $A$ in $\text{Alg} L$ such that $AX_p = Y_p$ ($p = 1, 2, \cdots, n$), every $E$ in $L$ reduces $A$ and $A$ is a self-adjoint operator.

2. \[ \sup \left\{ \frac{\| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} y_{i} f_{k,i} \|}{\| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} x_{i} f_{k,i} \|} : m_i \in \mathbb{N}, l \leq n, E_{k,i} \in L \text{ and } f_{k,i} \in H \right\} < \infty \text{ and } x_{i,\sigma(i)}^{(q)} y_{i,\sigma(i)}^{(q)} \text{ is real for all } i = 1, 2, \cdots.\]

Proof. We assume the condition (2) holds. Then, without loss of generality, we may assume that \[ \sup \left\{ \frac{\| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} y_{i} f_{k,i} \|}{\| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} x_{i} f_{k,i} \|} : m_i \in \mathbb{N}, l \leq n, E_{k,i} \in L \text{ and } f_{k,i} \in H \right\} = 1. \]

Then \[ \| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} y_{i} f_{k,i} \| \leq \| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} x_{i} f_{k,i} \| \cdots \cdots (*). \]

Let \[ M = \left\{ \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} x_{i} f_{k,i} : l \leq n, m_i \in \mathbb{N}, E_{k,i} \in L \text{ and } f_{k,i} \in H \right\}. \]

Then $M$ is a linear manifold. Define $A : M \to H$ by $A(\sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} x_{i} f_{k,i}) = \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} y_{i} f_{k,i}$. Then $A$ is well-defined by (*). Extend $A$ to $M$ by continuity. Define $A|_{M^\perp} = 0$. Clearly $AX_p = Y_p$ and we know that $\| A|_{M} \| \leq 1$ ($p = 1, 2, \cdots, n$). For $m_i \in \mathbb{N}, l \leq n, E_{k,i} \in L$ and $f_{k,i} \in H$, \[ AE \left( \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} x_{i} f_{k,i} \right) = A \left( \sum_{i=1}^{l} \sum_{k=1}^{m_i} E E_{k,i} x_{i} f_{k,i} \right) = \sum_{i=1}^{l} \sum_{k=1}^{m_i} E E_{k,i} y_{i} f_{k,i}, \]
\[ EA \left( \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_{i} f_{k,i} \right) = E \left( \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} Y_{i} f_{k,i} \right) = \sum_{i=1}^{l} \sum_{k=1}^{m_i} E E_{k,i} Y_{i} f_{k,i}. \]

For every \( g \) in \( \overline{\mathcal{M}} \), \( E A g = E 0 = 0 \) and \( A E g = 0 \) since \( \langle E g, \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_{i} f_{k,i} \rangle = \langle g, \sum_{i=1}^{l} \sum_{k=1}^{m_i} E E_{k,i} X_{i} f_{k,i} \rangle = 0 \). So every \( E \) in \( \mathcal{L} \) reduces \( A \). Therefore, \( A \) is diagonal. Let \( A = (a_{ii}) \). Since \( A X_p = Y_p, y_{ij}^{(p)} = a_{ii} x_{ij}^{(p)} \) for all \( i, j \), and \( p = 1, 2, \cdots, n \). Since \( x_{i,\sigma(i)}^{(q)} y_{i,\sigma(i)}^{(q)} \) is real for all \( i = 1, 2, \cdots, n \), \( A \) is a self-adjoint operator.

Conversely, if the condition (1) holds, then \( E A X_i = A E X_i = E Y_i \) for every \( E \) in \( \mathcal{L} \) \( (i = 1, 2, \cdots, n) \). So \( A E X_i = E Y_i \) for every \( E \) in \( \mathcal{L} \) and every \( f \) in \( \mathcal{H} \) \( (i = 1, 2, \cdots, n) \). Thus \( A (\sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_{i} f_{k,i}) = \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} Y_{i} f_{k,i} \), for \( m_i \in \mathbb{N}, l \leq n, E_{k,i} \in \mathcal{L} \) and \( f_{k,i} \in \mathcal{H} \). So

\[
\| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} Y_{i} f_{k,i} \| \leq \| A (\sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_{i} f_{k,i}) \| \\
\leq \| A \| \| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_{i} f_{k,i} \|.
\]

If \( \| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_{i} f_{k,i} \| \neq 0 \), then

\[
\frac{\| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} Y_{i} f_{k,i} \|}{\| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_{i} f_{k,i} \|} \leq \| A \|.
\]

Hence, \( \sup \left\{ \left\| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} Y_{i} f_{k,i} \right\|: l \leq n, m_i \in \mathbb{N}, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H} \right\} \leq \| A \| \).

Since every \( E \) in \( \mathcal{L} \) reduces \( A \), \( A \) is diagonal. Let \( A = (a_{ii}) \). Since \( A X_p = Y_p, y_{ij}^{(p)} = a_{ii} x_{ij}^{(p)} \) for all \( i, j \), and \( p \). Since \( A \) is a self-adjoint operator, \( x_{i,\sigma(i)}^{(q)} y_{i,\sigma(i)}^{(q)} \) is real for all \( i = 1, 2, \cdots \).

Theorem 3. Let \( X_p = (x_{ij}^{(p)}) \) and \( Y_p = (y_{ij}^{(p)}) \) be in \( \mathcal{B}(\mathcal{H}) \) \( (p = 1, 2, \cdots) \) such that \( x_{i,\sigma(i)}^{(q)} \neq 0 \) for some fixed \( q \) and for all \( i \). Then the following statements are equivalent.
There exists an operator $A$ in $\text{Alg}\mathcal{L}$ such that $AX_p = Y_p \ (p = 1, 2, \cdots)$, every $E$ in $\mathcal{L}$ reduces $A$ and $A$ is a self-adjoint operator.

(2) \[ \sup \left\{ \frac{\| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} Y_{i} f_{k,i} \|}{\| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_{i} f_{k,i} \|} : m_i, l \in \mathbb{N}, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H} \right\} < \infty \]
and $x^{(q)}_{i,\sigma(i)}(q)$ is real for all $i = 1, 2, \cdots$.

Proof. If \[ \sup \left\{ \frac{\| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} Y_{i} f_{k,i} \|}{\| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_{i} f_{k,i} \|} : m_i, l \in \mathbb{N}, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H} \right\} = 1. \]
Then $\mathcal{M} = \left\{ \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_{i} f_{k,i} : m_i, l \in \mathbb{N}, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H} \right\}$.

Let $\mathcal{M}$ is a linear manifold. Define $A : \mathcal{M} \rightarrow \mathcal{H}$ by $A(\sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_{i} f_{k,i}) = \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} Y_{i} f_{k,i}$. Then $A$ is well-defined by (1). Extend $A$ to $\overline{\mathcal{M}}$ by continuity. Define $A|_{\overline{\mathcal{M}}^\perp} = 0$. Clearly $AX_p = Y_p$ and we know that $\|A|_{\overline{\mathcal{M}}} \| \leq 1 \ (p = 1, 2, \cdots)$.

$$AE \left( \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_{i} f_{k,i} \right) = A \left( \sum_{i=1}^{l} \sum_{k=1}^{m_i} EE_{k,i} X_{i} f_{k,i} \right) = \sum_{i=1}^{l} \sum_{k=1}^{m_i} EE_{k,i} Y_{i} f_{k,i}$$

and

$$EA \left( \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} X_{i} f_{k,i} \right) = E \left( \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i} Y_{i} f_{k,i} \right) = \sum_{i=1}^{l} \sum_{k=1}^{m_i} EE_{k,i} Y_{i} f_{k,i}.$$
For every $g$ in $\mathcal{M}^\perp$, $EA(g) = E(0) = 0$ and $AE(g) = 0$ since

$$\langle Eg, \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i}X_{i}f_{k,i} \rangle = \langle g, \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i}X_{i}f_{k,i} \rangle = 0.$$ 

So every $E$ in $\mathcal{L}$ reduces $A$. Therefore, $A$ is diagonal. Let $A = (a_{ii})$. Since $AX_p = Y_p$, $y_{ij}^{(p)} = a_{ii}x_{ij}^{(p)}$ for all $i$, $j$, and $p = 1, 2, \cdots$. Since $x_{i,\sigma(i)}^{(q)}y_{i,\sigma(i)}^{(q)}$ is real for all $i = 1, 2, \cdots$, $A$ is a self-adjoint operator.

Conversely, if $AX_i = Y_i$, then $EAX_i = AEY_i$ for every $E$ in $\mathcal{L}$ ($i = 1, 2, \cdots$). So $AEY_i f = EY_i f$ for every $E$ in $\mathcal{L}$ and every $f$ in $\mathcal{H}$ ($i = 1, 2, \cdots$). Thus $A(\sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i}X_{i}f_{k,i}) = \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i}X_{i}f_{k,i}$, $m_i, l \in \mathbb{N}$, $E_{k,i} \in \mathcal{L}$ and $f_{k,i} \in \mathcal{H}$. So

$$\| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i}X_{i}f_{k,i} \| \leq \| A(\sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i}X_{i}f_{k,i}) \| \leq \| A \| \| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i}X_{i}f_{k,i} \|.$$ 

If $\| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i}X_{i}f_{k,i} \| \neq 0$, then

$$\frac{\| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i}X_{i}f_{k,i} \|}{\| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i}X_{i}f_{k,i} \|} \leq \| A \|.$$ 

Hence sup \( \left\{ \frac{\| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i}X_{i}f_{k,i} \|}{\| \sum_{i=1}^{l} \sum_{k=1}^{m_i} E_{k,i}X_{i}f_{k,i} \|} : l, m_i \in \mathbb{N}, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H} \right\} \leq \| A \|.$$

Since every $E$ in $\mathcal{L}$ reduces $A$, $A$ is diagonal. Let $A = (a_{ii})$. Since $AX_p = Y_p$, $y_{ij}^{(p)} = a_{ii}x_{ij}^{(p)}$ for all $p$, $i$, and $j$. Since $A$ is a self-adjoint operator, $x_{i,\sigma(i)}^{(q)}y_{i,\sigma(i)}^{(q)}$ is real for all $i = 1, 2, \cdots$. \[\Box\]

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