PERMUTATIONS WITH PARTIALLY
FORBIDDEN POSITIONS

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Abstract. In this paper we consider the enumeration problem of permutations with partially forbidden positions, generalizing the notion of permutations with forbidden positions. As an alternative approach to this problem, we investigate the permanent maximization problem over some classes of $(0,1)$-matrices which have a given number of 1’s some of which lie in prescribed positions.

1. Introduction

The problem of enumerating ‘permutations with forbidden positions’ retains substantial importance in the theory of combinatorics. A typical example is the well-known derangement problem. A derangement of $\{1, 2, \cdots, n\}$ is a permutation $\sigma$ of $\{1, 2, \cdots, n\}$ with the property that $\sigma(i) \neq i$ for all $i = 1, 2, \cdots, n$. The problem is to find the number of derangements. This kind of problem can be converted into one of the evaluation of the permanent of certain $(0,1)$-matrix with 0’s in the ‘forbidden’ positions and 1’s elsewhere. For a matrix $A = [a_{ij}]$, the permanent of $A$, $\text{per} A$, is defined by

$$\text{per} A = \sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)},$$

where $S_n$ stands for the symmetric group on $\{1, 2, \cdots, n\}$. The number $d_n$ of derangements of $\{1, 2, \cdots, n\}$, the $n$-th derangement number, is equal to the permanent of $J_n - I_n$, where $J_n$ and $I_n$ denote the all 1’s
matrix of order \( n \) and the identity matrix of order \( n \) respectively. It is well known that

\[
d_n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}.
\]

The derangement problem can be generalized to a problem of enumerating permutations \( \sigma \) of \( \{1, 2, \cdots, n\} \) satisfying \( \sigma(i) \neq i \) for at least \( k \) of the \( i \)'s in \( \{1, 2, \cdots, n\} \), where \( k \) is an integer such that \( 1 \leq k \leq n \). The corresponding permutation matrices are ‘partially’ forbidden to have their 1’s in the main diagonal in the sense that they are allowed to have up to \( n - k \) 1’s in the main diagonal. It is easy to see that the number we are looking for in this problem is equal to

\[
\sum_{i=0}^{n-k} \binom{n}{i} d_{n-i}.
\]

In this paper, we consider a problem of enumerating the permutations with partially forbidden positions for some other settings of interest.

Let \( n \) be a positive integer and \( S = [s_{ij}] \) be a \((0,1)\)-matrix of order \( n \). Let \( d \) be an integer such that \( 0 \leq d \leq n^2 - \#(S) \), where and in the sequel for a \((0,1)\)-matrix \( A \), \( \#(A) \) denotes the number of 1’s in \( A \). Let \( \mathcal{R}(S,d) \) denote the class of all \((0,1)\)-matrices \( A \) of order \( n \) such that \( A \geq S \) and \( \#(A - S) = d \), where and in the sequel, \( A \geq S \) (resp. \( A \leq S \)) means that every entry of \( A \) is bigger (resp. less) than or equal to the corresponding entry of \( S \). If \( d = 0 \), then \( \mathcal{R}(S,d) \) consists of the matrix \( S \) only, and the number of permutations \( \sigma \) of \( \{1, 2, \cdots, n\} \) such that \( \sigma(i) \neq j \) whenever \( s_{ij} = 0 \) equals \( \text{per} S \) as in the case of derangement problem. We are interested in the following problem:

(i) What is the maximum value of the permanent function over the class \( \mathcal{R}(S,d) \)?

(ii) At which matrices in \( \mathcal{R}(S,d) \) is this value achieved?

This problem for \( S = O \) was investigated by Brualdi, Goldwasser and Michael[1]. Specifically they determined an upper bound for the permanent of a matrix in \( \mathcal{R}(O,d) \) for a given \( d \), and they determined all matrices in \( \mathcal{R}(O,d) \) with maximum permanent for \( d \) with \( n \leq d \leq 2n \) and for \( d \) with \( n^2 - 2n \leq d \leq n^2 \).

An \( n \times n \) \((0,1)\) matrix \( A = [a_{ij}] \) with row sum vector \((r_1, r_2, \cdots, r_n)\) is called a Ferrers’ matrix if \( r_1 \leq r_2 \leq \cdots \leq r_n \) and \( a_{ii} \geq a_{i2} \geq \cdots \geq a_{in} \)
for every \( i = 1, 2, \ldots, n \), and is denoted by \( F(r_1, r_2, \ldots, r_n) \). It is well known that, if \( r_i \geq i \) for all \( i = 1, 2, \ldots, n \), then

\[
\text{per} F(r_1, r_2, \ldots, r_n) = \prod_{i=1}^{n} (r_i - i + 1) \tag{1}
\]

[3]. For integers \( n, k \) with \( 0 \leq k \leq n - 1 \), let \( F(n, k) = [f_{ij}] \) be the \( n \times n \) (0,1) matrix defined by \( f_{ij} = 1 \) if and only if \( j < i + k \). For example \( F(n, 1) = \Delta_n \) where

\[
\Delta_n = \begin{bmatrix}
1 & 0 & \cdots & 0 & 0 \\
1 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \cdots & 1 & 0 \\
1 & 1 & \cdots & 1 & 1
\end{bmatrix}
\]

is the \( n \times n \) lower triangular (0,1)-matrix with \((n^2+n)/2\) 1’s in and under the main diagonal positions, and \( F(n, 2) \) is the \( n \times n \) lower Hessenberg matrix

\[
A = \begin{bmatrix}
1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
1 & 1 & 1 & 1 & \cdots & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1
\end{bmatrix}.
\]

Let \( d \) be an integer such that \( 0 \leq d \leq n^2 - \#(F(n, k)) \). In this paper, we maximize the permanent over \( \mathcal{R}(F(n, k), d) \) for the case that \( n \) is sufficiently large compared with \( d \), and for the case that \( d \leq 2 \). The problem concerning the maximization and minimization of spectral radii of matrices in the class \( \mathcal{R}(\Delta_n, d) \) was investigated by Brualdi and Hwang [2]. By (1), it follows that

\[
\text{per} F(n, k) = k^{n-k}n! \tag{2}
\]
2. Maximum permanent over $\mathcal{R}(F(n,k), d)$

For a matrix $A$ of order $n$ and for $\alpha, \beta \subset \{1, 2, \ldots, n\}$, let $A(\alpha|\beta)$ denote the matrix obtained from $A$ by deleting all rows in $\alpha$ and all columns in $\beta$, and let $A[\alpha|\beta] = A(\overline{\alpha}|\overline{\beta})$, where $\overline{\alpha}$ and $\overline{\beta}$ stand for the complements of $\alpha$ and $\beta$ relative to $\{1, 2, \ldots, n\}$ respectively. The following lemma is well known. The proof we give here is a little bit simpler than that in [4, p.19]. For integers $r$, $n$ with $0 \leq r \leq n$, let $Q_{r,n}$ denote the set of all $r$-subsets of $\{1, 2, \ldots, n\}$.

**Lemma 1** ([4, p. 18]). If $A, B$ are matrices of order $n$, then

\begin{equation}
\text{per}(A + B) = \sum_{r=0}^{n} \sum_{\alpha, \beta \in Q_{r,n}} \text{per} A[\alpha|\beta] \text{per} B[\overline{\alpha}|\overline{\beta}].
\end{equation}

**Proof.** Let $A = [a_{ij}]$, $B = [b_{ij}]$, and

\begin{equation}
f(x) = \text{per}(Ax + B) = \sum_{r=0}^{n} c_r x^r.
\end{equation}

In the expansion of

\[
\sum_{\sigma \in S_n} (a_{1\sigma(1)}x + b_{1\sigma(1)})(a_{2\sigma(2)}x + b_{2\sigma(2)}) \cdots (a_{n\sigma(n)}x + b_{n\sigma(n)}),
\]

the coefficient $c_r$ of $x^r$ clearly equals

\[
\sum_{\alpha, \beta \in Q_{r,n}} \text{per} A[\alpha|\beta] \text{per} B[\overline{\alpha}|\overline{\beta}],
\]

and formula (3) follows by plugging $x = 1$ into (4).

Let $L_k$ denote the back diagonal permutation matrix

\[
g = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & 0 & 1 \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
0 & 0 & 0 & \cdots & 1 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 1 & \cdots & 0 & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
1 & 0 & 0 & \cdots & 0 & 0 & 0
\end{bmatrix}
\]
of order $k$, and for an integer $d$ with $0 \leq d \leq (n - k)/2$. Let $\Gamma_d$ denote
the $n \times n$ matrix
\[
\begin{bmatrix}
O & \mathbb{L}_d \\
O & O
\end{bmatrix}.
\]

We make a rough guess that the maximum value of the permanent over
the class $\mathcal{R}(F(n, k), d)$ is achieved at a matrix with the ‘additional’ $d$ 1’s
being placed close to the back diagonal. From now on in the sequel, let
$E_{ij}$ denote the square matrix of suitable order all of whose entries are 0
except for the $(i, j)$-entry which is 1.

**Lemma 2.** Let $X = [x_{ij}]$ be an $n \times n$ Ferrers matrix with $\Delta_n \leq X$
whose row sum vector and column sum vector are $(r_1, r_2, \cdots, r_n)$ and
$(c_1, c_2, \cdots, c_n)$ respectively, and let $p, q$ be integers with $1 \leq p < q < n$
such that $x_{pq} = 0$. Let $A = X + E_{1q} + E_{pn}$, $B = X + E_{1n} + E_{pq}$. Then
\[
\text{per } A \leq \text{per } B
\]
where the inequality is strict unless $r_1 = r_2 = \cdots = r_p$
and $c_q = c_{q+1} = \cdots = c_n$.

**Proof.** By Lemma 1, we have
\[
\text{per } A = \text{per } X + \text{per } X(1|q) + \text{per } X(p|n) + \text{per } X(1, p|q, n),
\]
\[
\text{per } B = \text{per } X + \text{per } X(1|n) + \text{per } X(p|q) + \text{per } X(1, p|q, n).
\]
Let $Y = X[p + 1, \cdots, q|p, \cdots, q - 1]$ and let
\[
a = r_2(r_3 - 1) \cdots (r_p - p + 2),
\]
\[
a' = r_1(r_2 - 1) \cdots (r_{p-1} - p + 2),
\]
\[
b = c_{n-1}(c_{n-2} - 1) \cdots (c_q - (n - q - 1)),
\]
\[
b' = c_n(c_{n-1} - 1) \cdots (c_{q+1} - (n - q - 1)).
\]
Then $a \geq a'$ with equality if and only if $r_1 = r_2 = \cdots = r_p$, and $b \geq b'$
with equality if and only if $c_q = c_{q+1} = \cdots = c_n$. Now by the formula
\( (1) \)

\[
\text{per} X(1|n) = \text{per} X[2, \ldots, n|1, \ldots, n-1] = r_2 \text{per} X[3, \ldots, n|2, \ldots, n-1] = r_2(r_3 - 1) \text{per} X[4, \ldots, n|3, \ldots, n-1] \\
\vdots
= r_2(r_3 - 1) \cdots (r_p - p + 2) \text{per} X[p + 1, \ldots, n|p, \ldots, n-1] = a \text{ per} X[p + 1, \ldots, n|p, \ldots, n-1].
\]

On the other hand,

\[
\text{per} X(1|n) = c_{n-1} \text{per} X[p + 1, \ldots, n|p, \ldots, n-1]\]

\[
= c_{n-1}(c_{n-2} - 1) \text{per} X[p + 1, \ldots, n - 2|p, \ldots, n-3] \\
\vdots
= c_{n-1}(c_{n-2} - 1) \cdots (c_q - (n - q - 1)) \\
\times \text{per} X[p + 1, \ldots, q|p, \ldots, q-1] = b \text{ per} Y.
\]

Thus \( \text{per} X(1|n) = ab \text{ per} Y \). Similarly we can show that \( \text{per} X(1|q) = a'b' \text{ per} Y \), \( \text{per} X(p|n) = a'b \text{ per} Y \) and \( \text{per} X(p|q) = a'b' \text{ per} Y \). Hence

\[
\text{per} B - \text{per} A = (ab + a'b' - ab' - a'b) \text{ per} Y \\
= (a - a')(b - b') \text{ per} Y \\
\geq 0,
\]

where equality holds if and only if either \( a = a' \) or \( b = b' \). Thus the inequality is strict unless \( r_1 = r_2 = \cdots = r_p \) and \( c_q = c_{q+1} = \cdots = c_n \).

Let \( \text{Max}(S, d) \) denote the set of all matrices \( A \in \mathcal{R}(S, d) \) such that \( \text{per} A \geq \text{per} X \) for all \( X \in \mathcal{R}(S, d) \).

**Theorem 3.** Let \( c, d \) and \( n \) be positive integers such that \( n \geq c + 2d \). If \( d \) equals 1 or 2, then \( F(n, c) + \Gamma_d \) is the unique matrix in \( \text{Max}(F(n, c), d) \).
Proof. Let $G = F(n, c)$. By Lemma 2, it is straightforward that $\text{Max}(F(n, c), 1)$ consists of the single matrix $G + \Gamma_1$. To prove the theorem for the case $d = 2$, let $A \in \text{Max}(F(n, c), 2)$ and let $A_1 = G + E_{1,n} + E_{2,n}$, $A_2 = G + \Gamma_2$. Then by Lemma 2 again and by taking flip along the back diagonal, if necessary, we may assume that $A = A_1$ or $A = A_2$. By Lemma 1,

$$\begin{align*}
\text{per}A_1 &= \text{per}G + \text{per}G(1|n) + \text{per}G(2|n), \\
\text{per}A_2 &= \text{per}G + \text{per}G(1|n) + \text{per}G(2|n - 1) + \text{per}G(1, 2|n - 1, n)
\end{align*}$$

so that

$$\text{per}A_2 - \text{per}A_1 = \text{per}G(2|n - 1) + \text{per}G(1, 2|n - 1, n) - \text{per}G(2|n).$$

Since

$$G(2|n - 1) = \begin{bmatrix} (L_{n-2}h_c)^T & 0 \\ F(n - 2, c + 2) & h_c \end{bmatrix},$$

$$G(1, 2|n - 1, n) = F(n - 2, c + 2)$$

and

$$G(2|n) = \begin{bmatrix} (L_{n-2}h_c)^T & 0 \\ F(n - 2, c + 2) & h_{c+1} \end{bmatrix},$$

where $h_k$ stands for the sum of the last $k$ columns of the identity matrix of order $n - 2$ for $k = c, c + 1$, we get, by formula (2), that

$$\begin{align*}
\text{per}G(2|n - 1) &= c^2 \text{per}F(n - 3, c + 1) = c^2(c + 1)^{n-c-4}(c + 1)!, \\
\text{per}G(1, 2|n - 1, n) &= (c + 2)^{n-c-4}(c + 2)! = (c + 2)^{n-c-3}(c + 1)! \\
\text{per}G(2|n) &= c \text{per}F(n - 2, c + 1) = c(c + 1)^{n-c-3}(c + 1)!,
\end{align*}$$

whence

$$\begin{align*}
\text{per}A_2 - \text{per}A_1 &= (c + 1)![c^2(c + 1)^{n-c-4} + (c + 2)^{n-c-3} - c(c + 1)^{n-c-3}] \\
&\geq (c + 1)![c^2 + (c + 2) - c(c + 1)] \\
&= 2(c + 1)!(c + 1)^{n-c-4} > 0.
\end{align*}$$

Thus it follows that $A_2$ is the matrix at which the permanent function attains at its maximum, and the proof is complete. $\square$
Lemma 4. Let $c$ and $d$ be fixed positive integers, and let $A \in \text{Max} (F(n,c),d)$. If $n$ is sufficiently large, then the $d \times d$ submatrix $A[1, \cdots, d|n-d+1, \cdots, n]$ in the upper right corner of $A$ is a permutation matrix.

Proof. Let $G = F(n,c)$ and $U = A - G$. Let $p$ be the largest integer less than or equal to $n$ such that the row $p$ of $U$ is not a zero vector, and let $q$ be the smallest integer less than or equal to $n$ such that the column $q$ of $U$ is not a zero vector. Let $U_0 = U[1, \cdots, p|q, \cdots, n].$ Then by Lemma 2, the matrix $U_0$ can not have a zero row or zero column. Note that $U_0$ is a permutation matrix if and only if $p = n - q + 1 = d$. We may assume that $p \leq n - q + 1$ by taking flip along the back diagonal if necessary. Suppose that $U_0$ is not a permutation matrix. Then $p < d$. There is no $k \times k$ permutation submatrix of $U$ if $k > p$. Let $k$ be an integer such that $1 \leq k \leq p$. Let $\alpha_k = \{1, \cdots, k\}$, $\beta_k = \{n - k + 1, \cdots, n\}$. Then for every $k$-subset $\alpha$ of $\{1, \cdots, p\}$ and every $k$-subset $\beta$ of $\{n - q + 1, \cdots, n\}$, $\text{per}(\alpha | \beta) \leq \text{per}(\alpha_k | \beta_k)$ because $G(\alpha | \beta) \leq G(\alpha_k | \beta_k)$. Since $G(\alpha_k | \beta_k) = F(n-k, k+c)$, we have, by formula (2), that

$$\text{per}(\alpha | \beta) \leq (k+c)^{n-2k-c}(k+c)! = \frac{(k+c)!}{(k+c)^{2k+c}(k+c)^n}.$$

Since the number of $k \times k$ permutation submatrices of $U$ does not exceed $\binom{d}{k}$, and since $\text{per} G = c^{n-c}c!$, we have from Lemma 1 that

$$\text{per} A \leq \sum_{k=0}^{p} \frac{(k+c)!}{(k+c)^{2k+c}(k+c)^n} \leq d^! \sum_{k=0}^{p} (k+c)^n.$$

On the other hand, letting $\alpha_0 = \{1, \cdots, d\}$, $\beta_0 = \{n-d+1, \cdots, n\}$, we have

$$\text{per}(G + \Gamma_d) \geq \text{per} G(\alpha_0 | \beta_0) \text{ per} \Gamma_d(\alpha_0 | \beta_0) = \text{per} G(\alpha_0 | \beta_0) = (d+c)^{n-2d-c}(d+c)! = \frac{(d+c)!}{(d+c)^{2d+c}(d+c)^n},$$

since $G(\alpha_0 | \beta_0) = F(n-d, d+c)$. Thus

$$\frac{\text{per} A}{\text{per}(G + \Gamma_d)} \leq \frac{d!(d+c)^{2d+c}}{(d+c)!} \sum_{k=0}^{p} \left( \frac{k+c}{d+c} \right)^n \leq \frac{d!(d+c)^{2d+c}}{(d+c)!} (p+1) \left( \frac{p+c}{d+c} \right)^n.$$
Since $p < d$, the above inequality tells us that $\text{per}A < \text{per}(G + \Gamma_d)$ for every sufficiently large $n$, which is impossible since $G + \Gamma_d \in R(G, d)$ and $A \in \text{Max}(G, d)$. Therefore it has to be that $U_0$ is a permutation matrix.

**Theorem 5.** Let $c$ and $d$ be fixed positive integers. If $n$ is sufficiently large, then the maximum permanent over $R(F(n, c), d)$ is achieved uniquely at the matrix $F(n, c) + \Gamma_d$.

**Proof.** We prove the theorem by induction on $d$. The case $d \leq 2$ is treated in Theorem 3, and the induction starts. As in Lemma 4, let $G = F(n, c), A \in \text{Max}(G, d)$, and $U = A - G = [u_{ij}]$. Then by Lemma 4, $U[1, \ldots, d|n - d + 1 \ldots, n]$ is a permutation matrix of order $d$. Note that any square submatrix of $U$ is a permutation matrix unless it has a zero row or zero column. We call a subset $\delta = \{(i_1, j_1), \ldots, (i_k, j_k)\}$ of $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ a $k$-diagonal of $U$ if $U[i_1, \ldots, i_k | j_1, \ldots, j_k]$ is a permutation matrix of order $k$. For a $k$-diagonal $\delta = \{(i_1, j_1), \ldots, (i_k, j_k)\}$ of $U$, let $G(\delta) = G(i_1, \ldots, i_k | j_1, \ldots, j_k)$. Let $\Delta(k, U)$ denote the set of all $k$-diagonals of $U$. By Lemma 1, we see that

$$\text{per}A = \text{per}G + \sum_{k=1}^{d} \sum_{\delta \in \Delta(k, U)} \text{per}G(\delta).$$

We first show that $u_{1n} = 1$. Suppose, on the contrary, that $u_{1n} = 0$. Then there exist integers $p, q$ with $1 < p \leq d$ and $n - d + 1 \leq q < n$ such that $u_{1q} = u_{pn} = 1$. Then, clearly, $u_{pq} = u_{1n} = 0$. Let $B = A - E_{1q} + E_{1n} + E_{pq} - E_{pn}$, and $V = B - G$. Then $B \in R(G, d)$, $V = U - E_{1q} + E_{1n} + E_{pq} - E_{pn}$, and

$$\text{per}B = \text{per}G + \sum_{k=1}^{d} \sum_{\delta \in \Delta(k, V)} \text{per}G(\delta).$$

A $\delta \in \Delta(k, U)$ belongs to exactly one of the following four cases;

- **case (i)**: $(1, q) \notin \delta$ and $(p, n) \notin \delta$,
- **case (ii)**: $(1, q) \in \delta$ and $(p, n) \notin \delta$,
- **case (iii)**: $(1, q) \notin \delta$ and $(p, n) \in \delta$,
- **case (iv)**: $(1, q) \in \delta$ and $(p, n) \in \delta$. 


Let \(\delta_0 = \delta - \{(1, q), (p, n)\}\), and let \(\delta' = \delta\) if \(\delta\) belongs to case (i), = \(\delta_0 \cup \{(1, n)\}\) if \(\delta\) belongs to case (ii), = \(\delta_0 \cup \{(p, q)\}\) if \(\delta\) belongs to case (iii), = \(\delta_0 \cup \{(1, n), (p, q)\}\) if \(\delta\) belongs to case (iv). Then

\[
\text{per}B - \text{per}A = \sum_{k=1}^{d} \sum_{\delta \in \Delta(k, U)} (\text{per}G(\delta') - \text{per}G(\delta)).
\]

If \(\delta\) belongs to case (i) or (iv), then \(\text{per}G(\delta') - \text{per}G(\delta) = 0\). So, by letting \(\Delta^*(k - 1, U)\) denote the set of all \((k - 1)\)-diagonals which do not contain one of \((1, q), (p, n)\), we have

\[
\sum_{k=1}^{d} \sum_{\delta \in \Delta^*(k-1, U)} (\text{per}G(\delta') - \text{per}G(\delta))
\]

which is positive by Lemma 2 because \(G(\delta_0)\) is a Ferrers matrix, contradicting the maximality of \(A\). Thus it is proved that \(u_{1n} = 1\). Now expanding \(\text{per}A\) along the first row, we have

\[
\text{per}A = c^2 \text{per}(F(n - 2, c) + U(1|n)) + \text{per}(F(n - 1, c + 1) + U(1, n)).
\]

By induction,

\[
\text{per}(F(n - 2, c) + U(1|n)) \leq \text{per}(F(n - 2, c) + \Gamma_{d-1})
\]

\[
\text{per}(F(n - 1, c + 1) + U(1|n)) \leq \text{per}(F(n - 1, c + 1) + \Gamma_{d-1})
\]

where any of the equalities holds if and only if \(U(1|n) = \Gamma_{d-1}\). Thus

\[
\text{per}A \leq c^2 \text{per}(F(n - 2, c) + \Gamma_{d-1}) + \text{per}(F(n - 1, c + 1) + \Gamma_{d-1})
\]

\[
= \text{per}(F(n, c) + \Gamma_{d}),
\]

with equality if and only if \(U(1|n) = \Gamma_{d-1}\). Since \(A \in \text{Max}(G, d)\), it must be that \(U(1|n) = \Gamma_{d-1}\) and hence that \(U(1|n) = \Gamma_d\), and the proof is complete. \(\square\)
3. Matrices in Max($\Delta_n, d$)

If $c = 1$, then $F(n, c) = \Delta_n$. The permanent maximization problem over the class $\mathcal{R}(\Delta_n, d)$ can be interpreted in terms of graph theoretic terminologies as follows. Let $D_n$ be the directed graph with $n$ nodes $1, 2, \ldots, n$ and $(n^2 + 2)/2$ arcs $(i, j)$, for all $i, j$ with $i \geq j$. Then $\Delta_n$ is the adjacency matrix of $D_n$. To $D_n$, we would like to introduce some new arcs $(i, j)$ with $i < j$. We call such an arc $(i, j)$ a down-going arc.

A spanning subgraph $H$ of a directed graph $D$ is called a 1-factor of $D$ if the in-degree and the out-degree in $H$ of each vertex equal 1. It is well known that the number of 1-factors of a directed graph $D$ is equal to the permanent of the adjacency matrix of $D$. The problem here is to determine the set of $d$ down-going arcs to be added to $D_n$ in order to maximize the number of 1-factors of the resulting directed graph.

We conjecture that the maximum permanent over $\mathcal{R}(\Delta_n, d)$ is achieved at the matrix $\Delta_n + \Gamma_d$ if $d \leq n/2$. In what follows we evaluate the permanent of $\Delta_n + \Gamma_d$ in terms of some numbers defined by a recurrence relation similar to that of binomial coefficients, and show that every matrix in $\text{Max}(\Delta_n, d)$ is fully indecomposable.

Let $p$ be a fixed nonnegative integer. For nonnegative integers $n, k$, let $\binom{n}{k}_p$ be a number defined by

(i) $\binom{n}{1}_p = 1$, $\binom{n}{n}_p = 1$ for all $n = 1, 2, \ldots$,

(ii) $\binom{n}{k}_p = \binom{n-1}{k-1}_p + kp \binom{n-1}{k}_p = 1$ for $k \leq n - 1$,

(iii) $\binom{n}{k}_p = 0$ if $k > n$.

Note that $\binom{n}{k}_0 = \binom{n}{k}$ and $\binom{n}{k}_1 = S(n, k)$, the Stirling number of the second kind. In the next theorem we give a formula for the permanent of the matrix $\Delta_n + \Gamma_d$ in terms of the numbers $\binom{n}{k}_2$.

**Theorem 6.** Let $d, n$ be integers such that $0 \leq 2d \leq n$.

(a) If $2d \leq n - 1$, then $\text{per}(\Delta_n + \Gamma_d) = \sum_{k=1}^{d+1} \binom{d+1}{k}_2 k^{n-2d}(k - 1)!$.

(b) If $2d = n$, then $\text{per}(\Delta_n + \Gamma_d) = \sum_{k=1}^{d} \binom{d}{k}_2 (k + 1)!$. 
Proof. For nonnegative integers \( n, k, r \) with \( k + r \leq n \), let \( f_r(n, k) = \text{per}(F(n, k) + \Gamma_r) \). We first prove the recurrence relation

\[
f_r(n, k) = k^2 f_{r-1}(n - 2, k) + f_{r-1}(n - 1, k + 1)
\]

for \( f_r(n, k) \). Let \( A = F(n, k) + \Gamma_r \). Then \( \text{per} A = \text{per}(A - E_{1n}) + \text{per} A(1|n) \). Clearly \( \text{per}(A - E_{1n}) = k^2 \text{per}(F(n - 2, k) + \Gamma_{r-1}) = k^2 f_{r-1}(n - 2, k) \). Since \( A(1|n) = F(n - 1, k + 1) + \Gamma_{r-1} \), we have \( \text{per} A(1|n) = f_{r-1}(n - 1, k + 1) \). Thus (5) follows. We see that \( \text{per}(\Delta_n + \Gamma_d) = f_d(n, 1) \).

We now show, for each integer \( r \) with \( 0 \leq r \leq \min\{d, (n - 1)/2\} \), that

\[
f_d(n, 1) = \sum_{k=1}^{r+1} \binom{r + 1}{k} f_{d-r}(n - 2r + k - 1, k)
\]

Clearly the equality (6) holds for \( r = 0 \). Suppose that (6) holds for \( r - 1 \). Then by induction and by the recurrence relation (5), we get

\[
f_d(n, 1) = \sum_{k=1}^{r} \binom{r}{k} f_{d-r+1}(n - 2r + k + 1, k)
\]
= \sum_{k=1}^{r} \left( \binom{r}{k} \right)^2 \left[ k^2 f_{d-r}(n - 2r + k - 1, k) \right.
\left. + f_{d-r}(n - 2r + k, k + 1) \right]
= \binom{r}{1}^2 f_{d-r}(n - 2r, 1) + k^2 \sum_{k=2}^{r} \left( \binom{r}{k} \right)^2 f_{d-r}(n - 2r + k - 1, k)
\left. + \sum_{k=1}^{r-1} \binom{r}{k}^2 f_{d-r}(n - 2r + k, k + 1) + \binom{r}{r}^2 f_{d-r}(n - r, r + 1) \right]
= \binom{r}{1}^2 f_{d-r}(n - 2r, 1)
\left. + \sum_{k=2}^{r} \left( k^2 \binom{r}{k}^2 + \binom{r}{k - 1}^2 \right) f_{d-r}(n - 2r + k - 1, k) \right]
\left. + \binom{r}{r}^2 f_{d-r}(n - r, r + 1) \right]
= f_{d-r}(n - 2r, 1)
\left. + \sum_{k=2}^{r} \binom{r+1}{k} f_{d-r}(n - 2r + k - 1, k) + f_{d-r}(n - r, r + 1) \right]
= \sum_{k=1}^{r+1} \binom{r+1}{k} f_{d-r}(n - 2r + k - 1, k),

completing the proof of (6). In case that \(2d \leq n - 1\), plugging \(r = d\) into (6), we get

\[
f_d(n, 1) = \sum_{k=1}^{d+1} \binom{d+1}{k} f_0(n - 2d + k - 1, k)
= \sum_{k=1}^{d+1} \binom{d+1}{k} k^{n-2d}(k - 1)!
\]

with the aid of formula (2). If \(2d = n\), then plugging \(r = d - 1\) into (6), we get
\[ f_d(n, 1) = \sum_{k=1}^{d} \binom{d}{k} f_1(n - 2d + k + 1, k) \]
\[ = \sum_{k=1}^{d} \binom{d}{k} f_1(k + 1, k) \]
\[ = \sum_{k=1}^{d} \binom{d}{k} (k + 1)! \]
since \( F(k + 1, k) + \Gamma_1 \) is the all 1’s matrix of order \( k + 1 \).

An \( n \times n \) matrix is called partly decomposable if it has an \( s \times (n - s) \) zero submatrix. A matrix which is not partly decomposable is called fully indecomposable. If \( c \geq 2 \), then the matrix \( F(n, c) \) is fully indecomposable. So, any matrix obtained from \( F(n, c) \) by replacing some of the 0’s with 1’s is fully indecomposable. However there are many partly decomposable \((0, 1)\)-matrices \( A \) with \( A \geq \Delta_n \). Nevertheless the set \( \text{Max}(\Delta_n, d) \) contains no partly decomposable matrices as we see in the following.

**Theorem 7.** Let \( d, n \) be integers such that \( 0 \leq 2d \leq n \). Then every matrix in \( \text{Max}(\Delta_n, d) \) is fully indecomposable.

**Proof.** Let \( A = [a_{ij}] \in \text{Max}(\Delta_n, d) \). Suppose that \( A \) is partly decomposable so that there exist \( \alpha, \beta \subset \{1, 2, \cdots, n\} \) with \( |\alpha| + |\beta| = n \) such that \( A[\alpha|\beta] = O \). Let \( \alpha = \{i_1, i_2, \cdots, i_p\}, \beta = \{j_1, j_2, \cdots, j_q\} \) where \( p + q = n \) and \( i_1 < i_2 < \cdots < i_p, j_1 < j_2 < \cdots < j_q \). Let \( \gamma = \{p + 1, \cdots, n\} \). If \( i_p > p \), then \( A[\alpha|\beta] \) can not be a zero matrix. Thus it follows that \( i_p = p \), and hence that \( \alpha = \{1, 2, \cdots, p\} \). Similarly we can show that \( \beta = \{p + 1, \cdots, n\} \). Therefore \( A \) has the form
\[
A = \begin{bmatrix}
X & O \\
J & Y
\end{bmatrix},
\]
where \( J \) is the \((n - p) \times p \) matrix of all 1’s and \( X, Y \) are square matrices of orders \( p \) and \( n - p \) respectively. By taking flip along the back diagonal, if necessary, we can assume that \( X - \Delta_p \neq O \). So, there exist integers \( r, s \) with \( 1 \leq r < s \leq p \) such that \( a_{rs} = 1 \). Let \( B = A - E_{rs} + E_{rn} \). Then
\[
\text{per}B - \text{per}A = \text{per}A(r|n) - \text{per}A(r|s)
\]
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\[
= \sum_{i \neq r} (a_{is} - a_{in}) \text{per} A(r, i|s, n)
\geq (a_{ss} - a_{sn}) \text{per} A(r, s|s, n) > 0,
\]
contradicting the maximality of \( A \), and the proof is complete. \( \square \)

References


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