MAX-MIN CONTROLLABILITY OF
DELAY-DIFFERENTIAL GAMES IN HILBERT SPACES

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ABSTRACT. We consider a linear differential game described by the delay-differential equation in a Hilbert space $H$:

\[
\frac{d}{dt}x(t) = A_0x(t) + A_1x(t-h) + \int_{-h}^{0} a(s)A_2x(t+s)ds + B(t)u(t) + C(t)v(t) \quad \text{a.e. } t > 0
\]

\[
x(0) = g^0, \quad x(s) = g^1(s) \in [-h, 0),
\]

where $g = (g^0, g^1) \in M_2 = H \times L_2([-h, 0]; Y)$, $u \in L_2^{ac}(\mathbb{R}^+; U)$, $v \in L_2^{ac}(\mathbb{R}^+; V)$, $U$ and $V$ are Hilbert spaces, and $B(t)$ and $C(t)$ are families of bounded operators on $U$ and $V$ to $H$, respectively. $A_0$ generates an analytic semigroup $T(t) = e^{tA_0}$ in $H$.

The control variables $g, u$ and $v$ are supposed to be restricted in the norm bounded sets \{ $g : \|g\|_{M_2} \leq \rho$ \}, \{ $u : \|u\|_{L_2([0,t]; U)} \leq \delta$ \} and \{ $v : \|v\|_{L_2([0,t]; V)} \leq \gamma$ \} ($\rho, \delta, \gamma \geq 0$). For given $x^0 \in H$ and a given time $t > 0$, we study $\epsilon$-approximate controllability to determine $x(\cdot)$ for a given $g$ and $v(\cdot)$ such that the corresponding solution $x(t)$ satisfies $\|x(t) - x^0\| \leq \epsilon$ ($\epsilon > 0$ : a given error).

0. Introduction

In the Euclidean space, various types of differential games of pursuit and evasion have been studied extensively (cf. Hájek[4]). Our main concern is to study max-min controllability problems in games theory, where we are concerned with selection of pursuer’s controls from an admissible set against evader’s controls. The max-min controllability has been investigated by Chan and Li[1] in the Euclidean space and in the Banach space, Park et al.[6] were studied in the case $A_0$ generates a $C_0$-semigroup and $A_1(\cdot)$ instead of $a(s)A_2$ in (*) is in $L_1([-h, 0]; \mathcal{L}(X))$. But we deal with the case that $A_0$ generates an analytic semigroup, $a(\cdot) \in L^2([-h, 0]; R)$. and $A_2 \in \mathcal{L}(Y, Y^*)$. Recently, this system has been studied by many authors[3,5].

Received February 12, 2000.
2000 Mathematics Subject Classification: 49J25, 49J35, 49K25, 49N05.
Key words and phrases: max-min controllability, evader’s control, minimal time.
Here controls are assumed to belong to some norm bounded constraint sets and in such constraint sets, we want to find these controls steering a given initial state to a desired state. In this paper, we study the existence of optimal solutions which are the minimum of control times and the minimum norm of controls for the delay-differential equation (*). We derive necessary and sufficient conditions for a max-min controllability problem in game theory.

1. Preliminaries

We give the description of a linear delay-differential game in a Hilbert space. Let $C$ and $R$ be the sets of complex and real numbers, respectively and let $R^+$ be the set of non-negative numbers. Let $\Omega$ be bounded smooth on $R^n$ and $Y = H_0^1(\Omega), H = L^2(\Omega)$. The norms of $H, Y$ and the inner product of $H$ are denoted by $\| \cdot \|, \| \cdot \|$ and $\langle \cdot, \cdot \rangle$ respectively. By identifying the antidual of $H$ with $H$ we may consider $Y \hookrightarrow \rightarrow H \equiv H^* \hookrightarrow \rightarrow Y^*$. The norm of the dual space $Y^*$ is denoted by $\| \cdot \|^*$. We consider a linear game described by an abstract delay-differential system(s) on $H$:

\begin{align}
\frac{dx(t)}{dt} &= A_0 x(t) + A_1 x(t-h) + \int_{-h}^0 a(s) A_2 x(t+s) ds \\
&\quad + B(t) u(t) + C(t) v(t), \text{ a.e. } t > 0.
\end{align}

(1.1)

where $g = (g^0, g^1) \in M_2 = H \times L_2([-h,0]; Y), u \in L^2_{loc}(R^+; U), v \in L^2_{loc}(R^+; V), \{ B(t) : t \geq 0 \} \subset L(U, H)$ is a strongly continuous family of bounded operators from $U$ into $H, \{ C(t) : t \geq 0 \} \subset L(V, H)$ is also a strongly continuous family of bounded operators from $V$ into $H, A_0$ generates an analytic semigroup $T(t) = e^{tA_0}$ both in $H$ and $Y^*$ and that $T(t) : Y^* \rightarrow Y$ for each $t > 0$ and $\eta$ is a stieltjes measure given by

\begin{align}
\eta(s) &= -\chi_{(-\infty,-h]}(s) A_1 - \int_s^0 a(\xi) d\xi A_2 : Y \rightarrow Y^*, s \in [-h,0],
\end{align}

(1.3)

where $\chi_{(-\infty,-h]}(\cdot)$ denotes the characteristic function of $(-\infty, -h]$. The delayed terms in (1.1) are written simply by $\int_{-h}^0 d\eta(s) x(t+s)$.

Let $a(x_1, x_2)$ be a bounded sesquilinear form defined in $Y \times Y$ satisfying Gårding’s inequality

\begin{align}
\text{Re } a(x, x) \geq c_0 |x|^2 - c_1 |x|^2,
\end{align}

(1.4)
where $c_0$ and $c_1$ are real constants. Let $A_0$ be the operator associated with the sesquilinear form

$$
(1.5) \quad \langle x_1, A_0 x_2 \rangle = -a(x_2, x_1), x_1, x_2 \in Y,
$$

where $\langle \cdot, \cdot \rangle_{Y, Y^*}$ denotes also the duality pairing between $Y$ and $Y^*$. The operator $A_0$ is bounded linear from $Y$ into $Y^*$. The realization of $A_0$ in $H$, which is the restriction of $A_0$ to the domain $\mathcal{D}(A_0) = \{ x \in Y : A_0 x \in H \}$ is also denoted by $A_0$.

Throughout this paper it is assumed that each $A_i(i = 1, 2)$ is bounded and linear from $Y$ to $Y^*$ (i.e. $A_i \in \mathcal{L}(Y, Y^*)$) such that $A_i$ maps $\mathcal{D}(A_0)$ endowed with the graph norm of $A_0$ to $H$ continuously. The real valued scalar function $a(s)$ is assumed to be $L^2$-integrable on $[-h, 0]$, that is $a(\cdot) \in L^2([-h, 0]; \mathbb{R})$. Let $W(t)$ be the fundamental solution of (s), which is a unique solution of the equation

$$
(1.6) \quad W(t) = \begin{cases} 
T(t) + \int_0^t T(t-s)\int_{-h}^0d\eta(\xi)W(\xi+s)ds, & t \geq 0, \\
0, & t < 0
\end{cases}
$$

i.e.

$$
(1.7) \quad W(t) = \begin{cases} 
T(t) + \int_0^t (A_1 W(s - h) + \int_{-h}^0 a(\sigma)A_2 W(\sigma + s)ds)ds, & t \geq 0 \\
0, & t < 0
\end{cases}
$$

Then $W(t) \in \mathcal{L}(H)$ for each $t \geq 0$ and $W(t)$ is strongly continuous in $R^+$ $[0, \infty)$ and $AW(t)$ and $\frac{d}{dt}W(t)$ are strongly continuous except at $t = nh, n = 0, 1, 2, \ldots$. Therefore we may assume that

$$
(1.8) \quad |W(t)| \leq M, t \geq 0, \text{ where } M \text{ is a constant.}
$$

The solution of (1.1) is expressed by

$$
(1.9) \quad x(t, g, u, v) = \begin{cases} 
W(t)g^0 + \int_{-h}^0 U_i(s)g^1(s)ds + \int_0^t W(t - s)B(s)u(s)ds + \int_0^t W(t - s)C(s)v(s)ds, & t \geq 0 \\
g^1(t), \text{a.e.} t \in [-h, 0),
\end{cases}
$$

where

$$
(1.10) \quad U_i(s) = W(t-s-h)A_1 + \int_{-h}^s W(t-s+\sigma)a(\sigma)d\sigma
$$

is well defined and is an element of $C(R^+; H)$.

The function $x(t) = x(t, g, u, v)$ is a unique solution of the integrated form of (1.1), (1.2) by $T(t)$. In this sense $x(t)$ is called the mild solution of the system(s). In the system(s), $u(t)$, $v(t)$ and $(g^0, g^1(s))$ are called a pursuer’s control, an evader’s control on forcing term and an evader’s control on initial data, respectively.
The state space $M_2 = H \times L_2([-h, 0]; Y)$ of the system $(s)$ is the product reflexive space with the norm
\[
\|g\|_{M_2} = (\|g^0\|^2 + \int_{-h}^{0} \|g^1\|^2 ds)^{\frac{1}{2}}, \quad g = (g^0, g^1) \in M_2.
\]
The dual space $M_2^*$ of $M_2$ is identified with the product space $H \times L_2([-h, 0]; Y^*) \equiv H \times L_2([-h, 0]; Y)^*$ via the duality pairing
\[
\langle g, f \rangle_{M_2} = \langle g^0, f^0 \rangle_{H} + \int_{-h}^{0} \langle g^1(s), f^1(s) \rangle_{Y^*} ds,
\]
where $g = (g^0, g^1) \in M_2$, $f = (f^0, f^1) \in M_2^*$ and $\langle \cdot, \cdot \rangle_{Y^*}$ denotes the duality pairing between $Y$ and $Y^*$. Here we note that the pairing $\langle \cdot, \cdot \rangle_{Y^*}$ is assumed to satisfy $\langle g^0, \alpha f^0 \rangle = \langle \bar{\alpha} g_0, f^0 \rangle$ for $\alpha \in \mathbb{C}$, $(g^0, f^0) \in H \times H$, $\bar{\alpha}$ being the conjugate of $\alpha$. We denote the norm in $Y^*$ by $|| \cdot ||_*$. For more detailed structural properties of the equations (1.1), (1.2) on the space $M_2$, we refer to [3].

2. Max-Min controllability

In this section, we study a max-min controllability problem which is noncooperative in the sense that against one evader’s controls, the other pursuer can select an appropriate control. For each $t > 0, \rho \geq 0, \gamma \geq 0$, we define constraint sets
\[
U^t_\delta = \{u \in L_2([0, t]; U) : ||u||_{2, [0, t]} = (\int_0^t |u(s)|^2_U ds)^{\frac{1}{2}} \leq \delta\},
\]
\[
V^t_\gamma = \{v \in L_2([0, t]; V) : ||v||_{2, [0, t]} = (\int_0^t |v(s)|^2_V ds)^{\frac{1}{2}} \leq \gamma\}
\]
and
\[
G_\rho = \{g \in M_2 : ||g||_{M_2} = (\|g^0\|^2 + \int_{-h}^{0} \|g^1(s)\|^2 ds)^{\frac{1}{2}} \leq \rho\}.
\]
The set $U^t_\delta, V^t_\gamma$ and $G_\rho$ are convex and closed in $L_2([0, t]; U), L_2([0, t]; V)$ and $M_2$, respectively. We put $Y^t_{\gamma, \rho} = G_\rho \times V^t_\gamma$ for evader’s constraint sets and define the reachable set $\mathcal{R}_t(Y^t_{\gamma, \rho})$ with respect to (i.e. w.r.t.) evader’s controls by
\[
\mathcal{R}_t(Y^t_{\gamma, \rho}) = \{x \in H : x = x(t; g, 0, v)\} \text{where}(g, v) \in Y^t_{\gamma, \rho}.
\]

**Lemma 2.1.** The set $\mathcal{R}_t(Y^t_{\gamma, \rho})$ is closed and convex for any $t > 0, \gamma \geq 0, \rho \geq 0$.  

Proof. It is clear that \( \mathcal{R}(Y_{\gamma,\rho}^t) \) is convex. We shall prove \( \mathcal{R}(Y_{\gamma,\rho}^t) \) is closed. Let \( x(t; g_n, 0, v_n) \) strongly converge to some \( x_0 \in H \) as \( n \to \infty \) for \( (g_n, v_n) \in Y_{\gamma,\rho}^t \). Then we have to prove that \( x_0 = x(t; g, 0, v) \) for some \( (g, v) \in Y_{\gamma,\rho}^t \). Since \( Y_{\gamma,\rho}^t \) is bounded in the reflexive product space \( M_2 \times L_2([0, t]; Y) \), there exists a subsequence (which we denote again by \( \{(g_n, v_n)\} \) of \( \{(g_n, v_n)\} \) weakly convergent to \( (g, v) \) (e.g. K. Yosida [7, p141]). Furthermore, by \( \|g\|_{M_2} \leq \liminf_{n \to \infty} ||g_n||_{M_2} \) and \( ||v||_{2,[0,t]} \leq \liminf_{n \to \infty} ||v_n||_{2,[0,t]} \) (e.g. [7, p120]). We see \( (g, v) \in Y_{\gamma,\rho}^t \).

Let \( x^* \in H \). Then by (1.8),

\[
\langle x(t; g_n, 0, v_n), x^* \rangle = \langle W(t) g_n^0 + \int_{-h}^t U_t(s) g_n^1(s) ds + \int_0^t W(t-s) C(s) v_n(s) ds, x^* \rangle
\]

\[
= \langle (g_0^0, g_1^1), (W(t)x^*, U_t^*(\cdot)x^*)) \rangle_{M_2} + \int_0^t \langle v_n(s), C^*(s) W^*(t-s)x^*) \rangle_V ds.
\]

Here it can be verified by the strong continuity of \( W(t) \) and \( C(t) \) and the equation (1.7) that \( (W^*(t)x^*, U_t^*(\cdot)x^*) \in M_2^* \) and \( C^*(t) W^*(t-s)x^* \in (L_2([0, t]; V))^* = L_2([0, t]; V^*) \). Since \( \{(g_n, v_n)\} \) is weakly convergent to \( (g, v) \) and \( x(t; g_n, 0, v_n) \) is strongly convergent to \( x_0 \), the above equality implies, by letting \( n \to \infty \), that

\[
\langle x_0, x^* \rangle = \langle (g^0, g^1), (W^*(t)x^*, U_t^*(\cdot)x^*)) \rangle_{M_2} + \int_0^t \langle v(s), C^*(s) W^*(t-s)x^*) \rangle_V ds
\]

\[
= \langle W(t) g^0 + \int_{-h}^t U_t(s) g^1(s) ds + \int_0^t W(t-s) C(s) v(s) ds, x^* \rangle
\]

since \( x^* \in H \) is arbitrarily chosen \( x_0 = W(t) g^0 + \int_{-h}^t U_t(s) g^1(s) ds + \int_0^t W(t-s) C(s) v(s) ds \), and hence \( x_0 \in \mathcal{R}(Y_{\gamma,\rho}^t) \). This completes the proof of Lemma 2.1.

\[
\square
\]

**Lemma 2.2.** ([6]) Let \( E \) and \( F \) be closed convex sets in \( X \). Then \( E \subseteq F \) if and only if

\[
\sup_{x \in E} \langle x, x^* \rangle \leq \sup_{x \in F} \langle x, x^* \rangle \quad \text{for all } x^* \in H.
\]

**Definition 2.1.** The system(s) is said to be max-min \((\delta, \gamma, \rho)\)-controllable on \([0, t]\) with respect to \( B(x^0; \epsilon) \) if each evader’s controls \((g, v) \in Y_{\gamma,\rho}^t\).
there exists a pursuer’s control \( u \in U^t \) such that \( x(t; g, u, v) \in B(x^0; \epsilon) \), where \( B(x^0; \epsilon) = \{ x \in X \mid |x - x^0| \leq \epsilon \} (\epsilon \geq 0) \).

Here \( x^0 \) is assumed to be a described state (a target point), and \( B(x^0; \epsilon) \) is a target set with error \( \epsilon \). Henceforth \( \langle \cdot, \cdot \rangle_U \) and \( \langle \cdot, \cdot \rangle_V \) denote the duality pairings between \( U \) and \( U^* \) and, \( V \) and \( V^* \), and \( | \cdot |_{U^*} \) and \( | \cdot |_{V^*} \) denote the norms in \( U^* \) and \( V^* \), respectively.

Using Lemma 2.1 and Lemma 2.2, we obtain the following result.

**Theorem 2.1.** The system(s) is max-min \((\delta, \gamma, \rho)\)-controllable on \([0, t]\) with respect to \( B(x^0; \epsilon) \) if and only if

\[
|\langle x^0, x^* \rangle| - \epsilon |x^*| \\
\leq \delta ||B^*(\cdot)W^*(t - \cdot)x^*||_{2,[0,t]} - \gamma ||C^*(\cdot)W^*(t - \cdot)x^*||_{2,[0,t]} - \rho ||(W^*(t)x^*, U^t_1(\cdot)x^*)||_{M^*_2} \text{ for each } x^* \in H,
\]

where

\[
||B^*(\cdot)W^*(t - \cdot)x^*||_{2,[0,t]} = (\int_0^t |B^*(s)W^*(t - s)x^*|_{H^*}^2 ds)^{\frac{1}{2}},
\]

\[
||C^*(\cdot)W^*(t - \cdot)x^*||_{2,[0,t]} = (\int_0^t |C^*(s)W^*(t - s)x^*|_{V^*}^2 ds)^{\frac{1}{2}},
\]

\[
||(W^*(t)x^*, U^t_1(\cdot)x^*)||_{M^*_2} = ||W^*(t)x^*||^2 + \int_{-h}^0 ||U^t_1(s)x^*||_{V^*}^2 ds)^{\frac{1}{2}}
\]

and

\[
U^t_1(s) = A^t_1W^*(t - s - h) + \int_{-h}^s a(\theta)W^*(t - s - \theta)d\theta \text{ a.e. } s \in [-h, 0).
\]

By (2.5) it is evident that the max-min \((\delta, \gamma, \rho)\)-controllability of the system(s) on \([0, t] \) w.r.t. \( B(x^0; \epsilon) \) implies the max-min \((\delta', \gamma', \rho')\)-controllability of(s) on \([0, t] \) w.r.t. \( B(x^0; \epsilon') \) for any \( \delta' \geq \delta, \rho' \leq \rho, \gamma' \leq \gamma \) and \( \epsilon' \geq \epsilon \).

**Proof.** For each \( t > 0 \), we define two operators

\[
B_t : L_2([0, t]; U) \to H \text{ and } Z_t : M_2 \times L_2([0, t]; V) \to H \text{ by}
\]

\[
B_t u = \int_0^t W(t - s)B(s)u(s)ds
\]

and

\[
Z_t (g, v) = W(t)g^0 + \int_{-h}^0 U^t_1(s)g^1(s)ds + \int_{-h}^0 W(t - s)C(s)v(s)ds
\]

respectively.
It is verified as in the proof of Lemma 2.1 that $B_t(U^t_\delta)$ and $Z_t(Y^{t}_{\gamma, \rho})$ are closed convex in $H$. By definition, the system(s) is max-min $(\delta, \gamma, \rho)$-controllable on $[0, t]$ with respect to $B(x^0; \epsilon)$ iff

$$
Z_t(Y^{t}_{\gamma, \rho}) \subset -B_t(U^t_\delta) + B(x^0; \epsilon).
$$

(2.12)

By Lemma 2.1, the set $Z_t(Y^{t}_{\gamma, \rho})$ is closed in $H$ and the set $-B_t(U^t_\delta) + B(x^0; \epsilon)$ is also closed. (In fact, it is obvious that $B_t(U^t_\delta) + B(x^0; \epsilon)$ is convex. Since both $B_t(U^t_\delta)$ and $B(x^0; \epsilon)$ are weakly closed and bounded, these sets are weakly compact (cf, [2, p.425]). Then the sum $B_t(U^t_\delta) + B(x^0; \epsilon)$ is weakly compact, and hence weakly closed. Therefore by the well known theorem (cf, [2, p.442]), $B_t(U^t_\delta) + B(x^0; \epsilon)$ is closed. Since both $Z_t(Y^{t}_{\gamma, \rho})$ and $-B_t(U^t_\delta) + B(x^0; \epsilon)$ are convex, then we can apply Lemma 2.2, to obtain that (2.12) is equivalent to

$$
\sup\{\langle Z_t(g, v), x^* \rangle; (g, v) \in Y^{t}_{\gamma, \rho}\}
$$

$$
\leq \sup\{\langle B_t u + y, x^* \rangle; u \in U^t_\delta, y \in B(x^0; \epsilon)\}
$$

for each $x^* \in H$.

By (2.11), we have

$$
\sup\{\langle Z_t(g, v), x^* \rangle; (g, v) \in Y^{t}_{\gamma, \rho}\}
$$

$$
= \sup\{(W(t)g^0 + \int_{-h}^{t} U_t(s)g^1(s)ds, x^*); (g^0, g^1) \in G_\rho\}
$$

$$
+ \sup\{\int_{0}^{t} W(t-s)C(s)v(s)ds, x^*\}; v \in V^{t}_{\gamma}\}
$$

$$
= \rho \sup\{(g^0, W^*(t)x^*) + \int_{-h}^{t} \langle g^1(s), U^*_t(s)x^* \rangle ds; \|(g^0, g^1)\|_{M_2} \leq 1\}
$$

$$
+ \gamma \sup\{\int_{0}^{t} \langle v(s), C^*(s)W^*(t-s)x^* \rangle v ds; \|v\|_{2, [0, t]} \leq 1\}
$$

$$
= \rho \|(W^*(t)x^*, U^*_t(\cdot)x^*)\|_{M_2^*} + \gamma \|C^*(\cdot)W^*(t-\cdot)x^*\|_{2, [0, t]}.
$$

On the other hand, by (2.10) the right side of (2.13) is calculated as follows;

$$
\sup\{\int_{0}^{t} W(t-s)B(s)u(s)ds, x^*\}; u \in U^t_\delta\}
$$

$$
+ \langle x^0, x^* \rangle + \sup\{(y, x^*); |y| \leq \epsilon\}
$$

$$
= \sup\{\int_{0}^{t} \langle u(s), B^*(s)W^*(t-s)x^* \rangle u ds; \|u\|_{2, [0, t]} \leq \delta\}
$$

$$
+ \langle x_0, x^* \rangle + \epsilon \sup\{(z, x^*); |z| \leq 1\}
$$

$$
= \delta \|B^*(\cdot)W^*(t-\cdot)x^*\|_{2, [0, t]} + \langle x_0, x^* \rangle + \epsilon |x^*|.
$$
Replacing \( x^* \) by \( -x^* \) in (2.15), we obtain condition (2.5). This completes the proof.

Next we consider the continuity of max-min controllability with respect to positive times \( t \), non-negative parameters \( \delta, \gamma, \rho, \epsilon \) and vectors \( x^0 \) in \( H \).

**Theorem 2.2.** Assume that the system \((s)\) is max-min \((\delta_n, \gamma_n, \rho_n)\)-controllable on \([0, t_n] \) w.r.t. \( B(x^0_n; \epsilon) \) for each \( n \geq 1 \). If

\[
\begin{align*}
(2.16) & \quad t_n \to t > 0, \ \delta_n \to \delta, \ \gamma_n \to \gamma, \ \rho \to \rho, \ \epsilon_n \to \epsilon \text{ in } \mathbb{R}^+ \\
(2.17) & \quad x^0_n \to x^0 \text{ weakly in } H \text{ as } n \to \infty,
\end{align*}
\]

then the system \((s)\) is max-min \((\delta, \gamma, \rho)\)-controllable on \([0, t] \) w.r.t. \( B(x^0; \epsilon) \).

Note that we require a weak convergence \( x^0_n \to x^0 \) not a strong one.

**Proof.** Since the system \((s)\) is max-min \((\delta_n, \gamma_n, \rho_n)\)-controllable on \([0, t_n] \) w.r.t. \( B(x^0_n; \epsilon_n) \), then by Theorem 2.1,

\[
\begin{align*}
(2.18) & \quad |\langle x^0_n, x^* \rangle| - \epsilon_n|\langle x^* \rangle| \\
& \quad \leq \delta_n||B^*(\cdot)W^*(t_n - \cdot)x^*||_{2,[0,t_n]} - \gamma_n||C^*(\cdot)W^*(t_n - \cdot)x^*||_{2,[0,t_n]} \\
& \quad - \rho_n||W^*(t_n)x^*, U^*_n(\cdot)||_{M^2_0}
\end{align*}
\]

for each \( x^* \in H \). Clearly by (2.17), we have

\[
(2.19) \quad |\langle x^0_n, x^* \rangle| \to |\langle x^0, x^* \rangle| \text{ as } n \to \infty.
\]

Let us set

\[
\begin{align*}
(2.20) & \quad F_1(t) = ||B^*(\cdot)W^*(t - \cdot)x^*||_{2,[0,t]}, \\
(2.21) & \quad F_2(t) = ||C^*(\cdot)W^*(t - \cdot)x^*||_{2,[0,t]}, \\
(2.22) & \quad F_3(t) = ||W^*(t - \cdot)x^*, U^*_t(\cdot)x^*)||_{M^2_0}.
\end{align*}
\]

Let \( T = \sup_{n \geq 1} t_n \) and \( I = [0, T] \). Since \( W(t) = 0 \) if \( t < 0 \), \( F_1(t_n) \) can be written as \( ||B^*(\cdot)W^*(t_n - \cdot)x^*)||_{2,I} \). By reflexivity of \( H \), \( W^*(t) \) is also strongly continuous on \( R^+ \) (cf. [3]), so that by (2.16)

\[
\begin{align*}
(2.23) & \quad \lim_{n \to \infty} W^*(t_n - s)x^* = \lim_{n \to \infty} W^*(t - s)x^* \text{ for all } s \in I
\end{align*}
\]

provided that \( t - s \neq 0 \). Since

\[
\begin{align*}
(2.24) & \quad |F_1(t_n) - F_1(t)| \\
& \quad \leq \left( \int_{I} |B^*(s)(W^*(t_n - s)x^* - W^*(t - s)x^*)|_{U^*_t}^2 \right)^{1/2} \\
& \quad \leq (\sup_{s \in I} ||B^*(s)||)(\int_{I} |W^*(t_n - s)x^* - W^*(t - s)x^*)|_{U^*_t}^2 \right)^{1/2}
\end{align*}
\]
and sup_{s \in I} ||B^*(s)|| = sup_{s \in I} ||B(s)|| is bounded by the strong continuity of B(\cdot) and the uniform boundedness principle, thus by applying the Lebesgue dominated convergence theorem, we have

(2.25) \quad F_1(t_n) \to F_1(t) \text{ as } n \to \infty.

By similar calculations, we can verify

(2.26) \quad F_2(t_n) \to F_2(t) \text{ as } n \to \infty.

Lastly we shall show

(2.27) \quad F_3(t_n) \to F_3(t) \text{ as } n \to \infty.

By the strong continuity of W^*(\cdot), we have

(2.28) \quad |W^*(t_n)x^*|^2 \to |W^*(t)x^*|^2 \text{ as } n \to \infty.

From (2.9) it follows that

(2.29) \quad U^*_i(t_n)x^* - U^*_i(s)x^*

= A_1^* W^*(t_n - s - h)x^* + \int_{-h}^{s} A_2^* a(\xi) W^*(t_n - s + \xi)x^* d\xi

- A_1^* W^*(t - s - h)x^* + \int_{-h}^{s} A_2^* a(\xi) W^*(t - s + \xi)x^* d\xi

a.e. \ s \in [-h,0].

We fix s such that the equality (2.29) holds. Then by (2.16) we have

(2.30) \quad A_1^* (W^*(t_n - s - h) - W^*(t - s - h)) \to 0 \text{ in } H,

provided that t - s - h \not= 0 ; and that

(2.31) \quad (W^*(t_n - s + \xi) - W^*(t - s + \xi))x^* \to 0 \text{ in } H

provided that t - s + \xi \not= 0 for each \ \xi \in [-h,s]. \ By (2.31), a(\cdot) \in L^2([-h,0];R) \text{ and the Lebesgue dominated convergence theorem, we see that}

(2.32) \quad | \int_{-h}^{s} A_2^* a(\xi)(W^*(t_n - s + \xi) - W^*(t - s + \xi))x^* d\xi |

\leq ||A_2|| \left( \int_{-h}^{0} |a(\xi)|^2 d\xi \right)^\frac{1}{2} \left( \int_{-h}^{s} |(W^*(t_n - s + \xi) - W^*(t - s + \xi))x^*|^2 d\xi \right)^\frac{1}{2}

\to 0 \text{ as } n \to \infty.

This implies, by (2.30) and (2.32), that for a.e. \ s \in [-h,0]

(2.33) \quad U^*_i(t_n)x^* \to U^*_i(s)x^* \text{ in } H \text{ as } n \to \infty.
Hence from (2.33), we have

\[
|\left(\int_{-h}^{0} |U_{t_n}(s)x^*|^2 ds\right)^\frac{1}{2} - \left(\int_{-h}^{0} |U_t^*(s)x^*|^2 ds\right)^\frac{1}{2}|
\leq \left(\int_{-h}^{0} \left|U_{t_n}(s)x^* - U_t^*(s)x^*\right|^2 ds\right)^\frac{1}{2} \to 0 \text{ as } n \to \infty,
\]

by applying the Lebesgue dominated convergence theorem again. Therefore we show (2.27). \qed

Now letting \( n \to \infty \) in (2.18) we reach the desired inequality (2.5) by (2.25)-(2.27). This proves, in view of Theorem 2.1, that the system(s) is max-min \((\delta, \gamma, \rho)\)-controllable on \([0, t]\) with respect to \(B(x^0; \epsilon)\).

### 3. Optimal value problems

In this section we study the existence of optimal solutions. Here, being optimal means the minimality of time interval \([0, t]\) over which we can control the system (Definition in section 2), the one of bound \(\delta\) of norms of pursuer’s controls, the one of error \(\epsilon\) for the target point \(x_0\) or the maximality of bounds \(\gamma, \rho\) of norms of evader’s controls. Throughout this section, it is assumed that the system \((s)\) is max-min \((\delta, \gamma, \rho)\)-controllable on \([0, t]\) with respect to \(B(x^0; \epsilon)\) for some \(\delta, \rho, \gamma, t, x^0\) and \(\epsilon\).

First we show the following theorem standing the existence of the minimal time interval \([0, t_f]\) on which max-min controllability is preserved.

**Theorem 3.1.** Let

\[
\pi_T = \{t' \in R^+ - \{0\}; \text{the system } (s) \text{ is max-min} \quad (\delta, \gamma, \rho) - \text{controllable on } [0, t'] \text{ w.r.t. } B(x^0; \epsilon)\}.
\]

Then \(\inf \pi_T = 0\) or there exists a minimal time \(t_f > 0\) such that

\[
t_f = \min \pi_T.
\]

In particular, if \(\inf \pi_T > 0\), then the system \((s)\) remains max-min \((\delta, \gamma, \rho)\)-controllable on \([0, t_f]\) w.r.t. \(B(x^0; \epsilon)\), where \(t_f\) is given by (3.2).

**Proof.** Obviously, \(t_f = \inf \pi_T\) exists. If \(t_f > 0\), let \(\{t_n\} \subset \pi_T\) be a sequence such that

\[
\lim_{n \to \infty} t_n = \inf \pi_T = t_f > 0.
\]

Then by (3.3) we can apply Theorem 2.2 to obtain the conclusion \(t_f \in \pi_T\).
Now we introduce the following subsets of $H$ in order to characterize the optimal values for various optimal value problems given below;

(3.4) \[ H_B = \{ x^* \in H; \| B^*(\cdot)W^*(t - \cdot)x^* \|_{2,[0,t]} = 1 \} \]

(3.5) \[ H_C = \{ x^* \in H; \| C^*(\cdot)W^*(t - \cdot)x^* \|_{2,[0,t]} = 1 \} \]

(3.6) \[ H_G = \{ x^* \in H; \| (W^*(t)x^*, U^*_t(\cdot)x^*) \|_{M^*_2} = 1 \}. \]

\[ \Box \]

**Theorem 3.2.** Let

(3.7) \[ \pi_D = \{ \delta' \in R^+; \text{the system } (s) \text{ is max-min} \]

\[ (\delta', \gamma, \rho) - \text{controllable on } [0,t] \text{ w.r.t. } B(x^0; \epsilon) \}. \]

Then there exists a minimal bound $\delta_f$ such that

(3.8) \[ \delta_f = \min \pi_D. \]

In particular, the system $(s)$ remains max-min $(\delta_f, \gamma, \rho)$-controllable on $[0,t]$ w.r.t. $B(x^0; \epsilon)$. Further if $H_B \neq \phi$, then $\delta_f$ is given by

(3.9) \[ \delta_f = \max \{ 0, \hat{\delta} \} \]

where

(3.10) \[ \hat{\delta} = \sup \{ |\langle x^0, x^* \rangle| + \gamma \| C^*(\cdot)W^*(t - \cdot)x^* \|_{2,[0,t]} \]

\[ + \rho \| (W^*(t)x^*, U^*_t(\cdot)x^*) \|_{M^*_2} - \epsilon |x^*|; x^* \in H_B \}; \]

and if $H_B = \phi$, then $\delta_f = 0$.

**Proof.** By Theorem 2.2, we can readily see the existence of $\min \pi_D$. Next we have to prove (3.9). To this end, setting $\delta_f' = \max \{ 0, \hat{\delta} \}$, we have only to prove $\delta_f' = \delta_f$. First we consider the case $H_B \neq \phi$. Then by the definition (3.10) of $\hat{\delta}$, $\delta_f'$ is finite. We shall show (3.9). Since $\delta_f \in \pi_D$, the following inequality holds for each $x^* \in H$:

(3.11) \[ |\langle x^0, x^* \rangle| - \epsilon |x^*| \]

\[ \leq \delta_f \| B^*(\cdot)W^*(t - \cdot)x^* \|_{2,[0,t]} - \gamma \| C^*(\cdot)W^*(t - \cdot)x^* \|_{2,[0,t]} \]

\[ - \rho \| (W^*(t)x^*, U^*_t(\cdot)x^*) \|_{M^*_2}. \]

Taking the supremum of (3.11) on the set $H_B$, we have $\delta \leq \delta_f$ by definition of $\hat{\delta}$. Hence $\delta_f' \leq \delta_f$. We will divide the proof into the two cases $\delta_f < 0$ and $\delta_f \geq 0$. 
First we assume \( \hat{\delta} > 0 \). In order to show the equality \( \delta_f' = \delta_f \), assume contrary that \( \delta_f' \neq \delta_f \), that is, \( \hat{\delta} \leq \delta_f \). Since \( \delta_f = \min \pi_D \), we have \( \hat{\delta} \notin \pi_D \) and hence by Theorem 2.1, there exists a nonzero vector \( x^*_S \in H \) such that

\[
\langle x^0, x^*_S \rangle - e|x^*_S| > \hat{\delta}||B^*(\cdot)W^*(t - \cdot)x^*_S||_{2, [0, t]} - \gamma||C^*(\cdot)W^*(t - \cdot)x^*_S||_{2, [0, t]}
- \rho||W^*(t)x^*_S, U^*_t(\cdot)x^*_S)||_{M^*_2}.
\]

This implies

\[
\langle x^0, x^*_S \rangle - e|x^*_S| + \gamma||C^*(\cdot)W^*(t - \cdot)x^*_S||_{2, [0, t]} + \rho||W^*(t)x^*_S, U^*_t(\cdot)x^*_S)||_{M^*_2}
> \hat{\delta}||B^*(\cdot)W^*(t - \cdot)x^*_S||_{2, [0, t]}.
\]

On the other hand, by substituting \( x^* = x^*_S \) in (3.11) we have

\[
\langle x^0, x^*_S \rangle - e|x^*_S| + \gamma||C^*(\cdot)W^*(t - \cdot)x^*_S||_{2, [0, t]} + \rho||W^*(t)x^*_S, U^*_t(\cdot)x^*_S)||_{M^*_2}
\leq \delta_f||B^*(\cdot)W^*(t - \cdot)x^*_S||_{2, [0, t]}.
\]

By (3.13) and (3.14), it follows that \( \delta_f||B^*(\cdot)W^*(t - \cdot)x^*_S||_{2, [0, t]} > 0 \). Since \( \delta_f > \hat{\delta} > 0 \), we have

\[
||B^*(\cdot)W^*(t - \cdot)x^*_S||_{2, [0, t]} > 0.
\]

Let

\[
y^*_S = x^*_S/||B^*(\cdot)W^*(t - \cdot)x^*_S||_{2, [0, t]},
\]

then we see easily that \( y^*_S \in H_B \) and

\[
\langle x^0, y^*_S \rangle - e|y^*_S| + \gamma||C^*(\cdot)W^*(t - \cdot)y^*_S||_{2, [0, t]} + \rho||W^*(t)y^*_S, U^*_t(\cdot)y^*_S||_{M^*_2} > \hat{\delta}.
\]

The inequality (3.17) contradicts the definition (3.10) of \( \hat{\delta} \). Thus, in the case of \( \hat{\delta} > 0 \), we see \( \delta_f' = \delta_f \).

Second we assume \( \hat{\delta} \leq 0 \). Then we can show for each \( x^* \in H \)

\[
\langle x^0, x^* \rangle - e|x^*| \leq -\gamma||C^*(\cdot)W^*(t - \cdot)x^*||_{2, [0, t]}
- \rho||W^*(t)x^*, U^*_t(\cdot)x^*)||_{M^*_2}.
\]

When \( x^* \) satisfies \( ||B^*(\cdot)W^*(t - \cdot)x^*||_{2, [0, t]} \neq 0 \), we set

\[
y^* = x^*/||B^*(\cdot)W^*(t - \cdot)x^*)||_{2, [0, t]}.
\]
Then $y^* \in H_B$, so that by (3.10) we have
\begin{equation}
\hat{\delta} \geq |\langle x^0, y^* \rangle| + \gamma ||C^* (\cdot) W^* (t - \cdot) y^* ||_{2,[0,t]}
+ \rho ||(W^*(t) y^*, U^*_t (\cdot) y^*)||_{M^*_2} - \epsilon |y^*|, 
\end{equation}
and hence, by $\hat{\delta} \leq 0$, we get
\begin{equation}
|\langle x^0, x^* \rangle| - \epsilon |x^*| 
\leq -\gamma ||C^* (\cdot) W^* (t - \cdot) x^* ||_{2,[0,t]} - \rho ||(W^*(t) x^*, U^*_t (\cdot) x^*)||_{M^*_2}.
\end{equation}
Therefore (3.18) holds true in this case. Lastly, when $x^*$ satisfies
\begin{equation}
||B^* (\cdot) W^* (t - \cdot) x^* ||_{2,[0,t]} = 0
\end{equation}
for each $x^* \in H$. This shows $\delta_f = 0$. Hence, the proof is completed.

Next we consider the case $H_B = \emptyset$. Then
\begin{equation}
||B^* (\cdot) W^* (t - \cdot) x^* ||_{2,[0,t]} = 0 \text{ for each } x^* \in H.
\end{equation}
Therefore we have by (2.5)
\begin{equation}
|\langle x^0, x^* \rangle| - \epsilon |x^*| \leq \gamma ||C^* (\cdot) W^* (t - \cdot) x^* ||_{2,[0,t]}
- \rho ||(W^*(t) x^*, U^*_t (\cdot) x^*)||_{M^*_2}
\end{equation}
for each $x^* \in H$. This shows $\delta_f = 0$. Hence, the proof is completed.

By analogous argument, we can verify the following theorem on the existence of minimal target error $\epsilon$.

**Theorem 3.3.** Let
\begin{equation}
\pi_E = \{ \epsilon' \in R^+; \text{the system(s) is max-min (\delta, \gamma, \rho) - controllable on } [0,t] \text{ w.r.t. } B(x^0, \epsilon') \}.
\end{equation}

Then there exists a minimal value $\epsilon_f$ such that
\begin{equation}
\epsilon_f = \min \pi_E.
\end{equation}
In particular, the system(s) remains max-min (\delta, \gamma, \rho)-controllable on $[0,t]$ w.r.t. $B(x^0, \epsilon_f)$. Further $\epsilon_f$ is given by
\begin{equation}
\epsilon_f = \max \{0, \hat{\epsilon} \}.
\end{equation}
where
\[
\hat{\epsilon} = \sup \{ |\langle x^0, x^* \rangle| + \gamma |C^*(\cdot) W^*(t - \cdot) x^*|_{2,[0,t]} \\
- \rho ||(W^*(t)x^*, U^*_t(\cdot)x^*)||_{M^*_2} \\
- \delta ||B^*(\cdot) W^*(t - \cdot) x^*||_{2,[0,t]} |x^*| = 1 \}.
\]

The next two theorems are related to the existence of maximal norm-bounds for the evader’s controls.

**Theorem 3.4.** Let
\[
\pi_C = \{ \gamma' \in R^+; \text{the system(s) is max-min } (\delta, \gamma', \rho) - \text{controllable on } [0, t] \text{ with respect to } B(x^0; \epsilon) \}.
\]
If \( H_C \neq \emptyset \) and \( \pi_C \) is bounded, then there exists a maximal value \( \gamma_f \) such that
\[
\gamma_f = \max \pi_C.
\]
In particular, the system(s) remains max-min \((\delta, \gamma_f, \rho)\)-controllable on \([0, t]\) with respect to \( B(x^0; \epsilon) \). Further in this case the maximal value \( \gamma_f \) is given by
\[
\gamma_f = \max \{0, \hat{\gamma}\},
\]
where
\[
\hat{\gamma} = \inf \{ \delta ||B^*(\cdot) W^*(t - \cdot) x^*||_{2,[0,t]} + \epsilon |x^*| \\
- |\langle x^0, x^* \rangle| - \rho ||(W^*(t)x^*, U^*_t(\cdot)x^*)||_{M^*_2}; x^* \in H_C \}.
\]
If \( H_C = \emptyset \) or \( \pi_C \) is unbounded, then \( \pi_C = R^+ \).

**Theorem 3.5.** Let
\[
\pi_R = \{ \rho' \in R^+; \text{the system (s) is max-min } (\delta, \gamma, \rho') - \text{controllable on } [0, t] \text{ w.r.t. } B(x^0; \epsilon) \}.
\]
If \( H_G \neq \emptyset \) and \( \pi_R \) is bounded, then there exists a maximal value \( \rho_f \) such that
\[
\rho_f = \max \pi_R.
\]
In particular, the system (s) remains max-min \((\delta, \gamma, \rho_f)\)-controllable on \([0, t]\) w.r.t \( B(x^0; \epsilon) \). Further in this case the maximal value \( \rho_f \) is given by
\[
\rho_f = \max \{0, \hat{\rho}\},
\]
where
\[
\hat{\rho} = \inf \{ \delta \| B^* (t - \cdot)x^* \|_{L_2[0,t]} + \| x^* \| \\
- \gamma \| C^* (t - \cdot)x^* \|_{L_2[0,t]} - | \langle x^0, x^* \rangle ; x^* \in H_G \}.
\]

If \( H_G = \phi \) or \( \pi_R \) is unbounded, then \( \pi_R = R^+ \).

**Proof of Theorem 3.4 and Theorem 3.5.** We can prove these theorems in a manner similar to Theorem 3.2.

**References**


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