ON SOME PROPERTIES OF BOUNDED $X^*$-VALUED FUNCTIONS

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Abstract. Suppose that $X$ is a Banach space with continuos dual $X^{**}, (\Omega, \Sigma, \mu)$ is a finite measure space. $f : \Omega \to X^*$ is a weakly measurable function such that $x^{**}f \in L_1(\mu)$ for each $x^{**} \in X^{**}$ and $T_f : X^{**} \to L_1(\mu)$ is the operator defined by $T_f(x^{**}) = x^{**}f$. In this paper we study the properties of bounded $X^*$-valued weakly measurable functions and bounded $X^*$-valued weak*-measurable functions.

1. Introduction

Suppose that $X$ is a Banach space with continuos dual $X^{**}, (\Omega, \Sigma, \mu)$ is a finite measure space. $f : \Omega \to X^*$ is a weakly measurable function such that $x^{**}f \in L_1(\mu)$ for each $x^{**} \in X^{**}$ and $T_f : X^{**} \to L_1(\mu)$ is the operator defined by $T_f(x^{**}) = x^{**}f$.

In this paper we study the properties of bounded $X^*$-valued weakly measurable functions and bounded $X^*$-valued weak*-measurable functions.

Throughout the paper $X$ will denote the unit ball of $X$ by $B_X$. An operator $T_f : X^{**} \to L_1(\mu)$ is said to be ($w^*$, norm)-continuous provided that net $T_f(x_j^{**})$ converges to $T_f(x^{**})$ in the norm topology of $L_1(\mu)$ whenever $(x_j^{**})$ is a net which converges to $x^{**}$ in the weak* topology of $X^{**}$.

A function : $(\Omega, \Sigma, \mu) \to X^*$ is weakly measurable if $x^{**}f$ is measurable for every $x^{**} \in X^{**}$. A function : $(\Omega, \Sigma, \mu) \to X^*$ is weak* measurable if $x f$ is measurable for every $x \in X$.

An operator $T_f : X^{**} \to L_1(\mu)$ which is defined by $T_f(x^{**}) = x^{**}f$ is weakly compact it the norm closure of $T_f(B_X^{**})$ is weakly compact. A subset $K$ of $L_1(\mu)$ is called uniformly integrable if $\lim_{\mu(E) \to 0} \int_E |f| d\mu = 0$ uniformly in $f \in K$.

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2. Main Theorems

**Theorem 1.** If \( f : \Omega \to X^* \) is bounded weakly measurable function, then \( f \) is \((w^*, \text{norm})\)-sequentially continuous.

**Proof.** If \( x_n^{**} \) converges to \( x^{**} \) in the weak* topology of \( X^{**} \), then \( x_n^{**} f \) converges to \( x^{**} f \) pointwise. Since \( x_n^{**} \) converges to \( x^* \) in the weak* topology of \( X^{**} \), by the principle of uniform boundedness \( \sup_{n \to \infty} ||x_n^{**}|| < \infty \) and by hypothesis there exists \( M > 0 \) such that \( \sup_{x \in \Omega} ||f|| < M \).

Since \( \|x_n^*\| \leq \sup \|x_n^{**}\| \leq M \), by Lebesgue’s bounded convergence theorem

\[
\lim_{n \to \infty} \|x_n^{**} f - x^{**} f\| = \lim_{n \to \infty} \int_{\Omega} |x_n^{**} f - x^{**} f| d\mu = 0
\]

Thus \( T_f \) is \((w^*, \text{norm})\)-sequentially continuous. \( \square \)

**Lemma.** A subset of \( L_1(\mu) \) be relatively weakly compact if and only if it is bounded and uniformly integrable.

**Proof.** Let \( K \subset L_1(\mu) \) be relatively weakly compact. Then \( K \) is bounded and if \( (f_n) \) is a sequence in \( K \), then \( (f_n) \) has a weakly convergent subsequence.

Hence there is a subsequence \( (f_{nj}) \) such that \( \lim_{j} \int_{\infty} E f_{nj} d\mu \) exists for all \( E \in \Sigma \).

It follows immediately that \( K \) is uniformly integrable.

Conversely, suppose \( K \) is bounded and uniformly integrable. Let \( (f_n) \) be a sequence in \( K \). Then there is a countable field \( F \) such that \( f_n \) is measurable relative to the \( \sigma \)-field \( \Sigma_1 \), generated by \( F \).

By diagonal procedure, select a subsequence \( (f_{nj}) \) such that \( \lim_{j} \int_{\infty} E f_{nj} d\mu = F(E) \) exists for all \( E \in F \).

Since \( K \) is uniformly integrable, there exists \( f \in L_1(\Sigma_1, \mu) \) such that

\[
\lim_{j} \int_{\infty} f_{nj} g d\mu = \int_{\infty} f g d\mu
\]

for each \( g \in L_{\infty}(\Sigma_1, \mu) \). Hence \( f_{nj} \to f \) is weakly in \( L_1(\Sigma_1, \mu) \), But \( f_{nj} \to f \) is weakly \( L_1(\mu) \), Hence \( K \) is relatively compact. \( \square \)

**Theorem 2.** If \( f : \Omega \to X^* \) is bounded weakly measurable function, then \( T_f : X^{**} \to L_1(\mu) \), is locally compact operator.
Proof. Since $f : \Omega \to X^*$ is bounded there exists a number $M$ such that $\sup \{|f(x)|; x \in \Omega\} \leq M$. If $x^{**}$ belongs to $B_{X^{**}}^*$, 
\[ ||T_f(x^{**})|| = \int_{\Omega} |x^{**}f|d\mu = \int_{\Omega} |f(x)|d\mu = M\mu(\Omega). \]
Hence $T_f(B_{X^{**}}^*)$ is norm bounded. If $\varepsilon > 0, \mu(B) < \frac{\varepsilon}{M}$ then 
\[ \int_{E} ||f||d\mu \leq M\mu(E) < \varepsilon \text{ and if } \mu(E) < \frac{\varepsilon}{M} \text{ and } x^{**} \in B_{X^{**}}^*, \]
\[ \int_{E} |T_f(x^{**})|d\mu = \int_{E} |x^{**}f|d\mu \leq \int_{E} ||f||d\mu < \varepsilon \]
Hence $T_f(B_{X^{**}}^*)$ is uniformly integrable. By Dunford theorem, $T_f(B_{X^{**}}^*)$ is relatively weakly compact. Therefore $T_f : X^{**} \to L(\mu)$ is weakly compact operator. □

**Theorem 3.** Suppose that $(\Omega, \Sigma, \mu)$ is a measure space, $f_n : \Omega \to X^*$ is bounded weak*-measurable for each $n \in N$, $\{f_n : n \in N\}$ is uniformly bounded and there is a real valued function $g_x$ on $\Omega$ such that $xf_n \to gx$ a.e.$[\mu]$. Then there is an $f : \Omega \to X^*$ such that $xf = gx$ a.e.$[\mu]$ for each $x \in X$.

Proof. Suppose the hypothesis are satisfied. Let $M_n$ be a sup$\{||f_n(x^*)|| : x^* \in X^*\}$, since $\{f_n : n \in N\}$ is uniformly bounded, $M = \sup M_n$.

Let $K_M(0)$ denote the closed ball of radius $M$ with center at the origin of $X^*$, the $K_M(0)$ is weak* compact and $(K_M(0), w^*)^\Omega$, there are a subset $(f_{n_k})$ of $(f_n)$ and a function $f : \Omega \to K_M(0)$ such that $(f_{n_k})$ converges to $f$ pointwise in the $w^*$-topology. But the $xf = gx$ a.e.$[\mu]$ for each $x \in X$. □

**References**


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