A REMARK ON THE CLIFFORD INDEX AND HIGHER ORDER CLIFFORD INDICES

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1. Introduction

In [KKM] it was seen that a linear series $g_{r+2r}$ computing the Clifford index $e$ of an algebraic curve $C$ is birationally very ample if $r \geq 3$ and $e \geq 3$. The purpose of this present note is to make further observations along the same lines. We also introduce the notion of higher order Clifford indices and make a few remarks on it.

We first fix some basic terminology and notations. $C$ always denote a smooth irreducible projective curve of genus $g \geq 4$. A $g_d^r$ on $C$ is a linear series of degree $d$ and (projective) dimension $r$ on $C$.

For a line bundle $L$ or a complete $g_d^r$ on $C$, we define the Clifford index $Cliff(L)$ of $L$ by

$$Cliff(L) = Cliff(g_d^r) = d - 2r = \deg L - 2h^0(C, L) + 2.$$ 

The Clifford index $e$ of $C$ is defined to be the non-negative integer

$$e := \min\{Cliff(L) \mid L \in \text{Pic}(C), h^0(C, L) \geq 2 \text{ and } h^1(C, L) \geq 2\}.$$ 

We say that a line bundle $L$ (or a complete $g_d^r$) on $C$ contributes to the Clifford index if $h^0(C, L) \geq 2$ and $h^1(C, L) \geq 2$ ($r \geq 1$ and $g - d + r - 1 \geq 1$).

We say that $L = g_d^r$ on $C$ computes the Clifford index of $C$ if $L$ contributes to the Clifford index and $d - 2r = e$; in this case $L = g_d^r$ is obviously base-point-free.

Throughout we work over the field of complex numbers.

* During the preparation of this work, both authors were almost fully supported by the Max-Planck-Institut für Mathematik. The second author is grateful to KOSEF for the travel grant to the MPI.

Received February 7, 1990.
2. Birationally very ample linear series

We start by giving a complement (the case \( r = 2 \)) to [KKM].

**Proposition 2.1.** Let \( g_d^2 \) on \( C \) compute the Clifford index \( e \) of \( C \). Then \( g_d^2 \) is birationally very ample unless \( C \) is a 2:1 covering of a smooth plane curve \( M \subseteq \mathbb{P}^2 \), degree(\( M \)) = \( e + 4/2 \).

**Proof.** Assume that \( |D| = g_d^2 = g_{e+4}^2 \) gives a \( k : 1 \) covering of \( \pi \) of a plane curve \( M \), \( k \geq 2 \). Fix \( p \in M \), let \( t \) be its multiplicity. Set \( E := \pi^*(p) \). Then \( |D - E| \) is a base-point-free \( g_1^{e+4-2k} \). Then by definition of the Clifford index, we must have \( k = 2 \) and \( t = 1 \).

**Proposition 2.2.** Let \( |D| = g_{2r+e+1}^2 \), \( r \geq 3 \), be a special linear series without base points on a curve \( C \) with Clifford index \( e \geq 1 \) such that \( r(K - D) \geq 1 \). Then \( |D| \) is birationally very ample.

**Proof.** Assume that \( |D| \) is not birationally very ample. Then \( |D| \) defines a morphism \( C \to \mathbb{P}' \) of degree \( m \geq 2 \) onto a curve \( C' \) in \( \mathbb{P}' \) of degree \( d' = \frac{2r+e+1}{n} \). We note that the induced complete linear series \( g_{d'}^r \) on \( C' \) is very ample since otherwise \( C' \) would admit a \( g_{d'-2}^{e-1} \) whence there were a \( g_{m(d'-2)}^{e-1} \) on \( C \) for which \( m(d' - 2) - 2(r - 1) = e - 2m + 3 < e \), a contradiction. Thus \( C' \) is a smooth non-degenerate linearly normal curve of degree \( d' \) in \( \mathbb{P}' \).

In particular \( d' \geq r \).

Let \( d' \geq r + 2 \). Then by the fact that any reduced irreducible and nondegenerate curve of degree at least \( r + 2 \) in \( \mathbb{P}' \), \( r \geq 3 \), has a \( r \)-secant-\((r - 2)\)-plane, we get a \( g_{d'-r}^1 \) on \( C' \) inducing a \( g_{m(d'-r)}^1 \) on \( C \). But \( m(d' - r) - 2 = e - 1 - r(m - 2) < e \), a contradiction. Thus \( C' \) in \( \mathbb{P}' \) is either a rational curve of degree \( r \) or an elliptic curve of degree \( r + 1 \). If \( C' \) is rational, \( C \) has a \( g_m^1 \) and we have

\[
m - 2 = \frac{e + 2r + 1}{r} - 2 = \frac{e + 1}{r} \geq e
\]

which is impossible. If \( C' \) is elliptic then \( C \) has a \( g_{2m}^1 \), and we have

\[
2m - 2 = 2 \frac{e + 2r + 1}{r + 1} - 2 \geq e
\]

which is only possible for \( m = 2 \) and \( e = 1 \).
In particular \( C \) must be a 2-sheeted cover of an elliptic curve. On the other hand, if \( e = 1 \) then \( C \) is either a smooth plane quintic or a trigonal curve. Since a smooth plane quintic cannot be elliptic-hyperelliptic, \( C \) must be trigonal. Because \( C \) also has a \( g^1_4 \), which is a pull-back of a \( g^1_2 \) on \( C' \), we have \( g(C) \leq (3 - 1) \cdot (4 - 1) = 6 \) by the Severi’s inequality. By Clifford’s theorem, we have

\[
1 \leq r(K - D) \leq \frac{2g - 2 - (2r + 2)}{2} - 1 = g - r - 3 \leq 3 - r
\]

which is contradictory to \( r \geq 3 \).

We take one step further to prove the following similar result.

**Proposition 2.3.** Let \( |D| = g^r_{2r+e+2}, r \geq 3 \) be a complete linear series without base point on a curve \( C \) with Clifford index \( e \geq 1 \) such that \( r(K - D) \geq 1 \). Then \( |D| \) is birationally very ample unless \( C \) is one of the following type:

(I) \( C \) is a triple cover of a curve of genus \( g' = 0 \) or 1.

(II) \( C \) is a double cover of a curve of genus \( g' = 2, 3, 4, 5 \) or 10.

**Proof.** Assume that \( |D| \) is not birationally very ample. Then \( |D| \) defines a morphism \( C \to \mathbf{P}^r \) of degree \( m \geq 2 \) onto a curve \( C' \) in \( \mathbf{P}^r \) of degree \( d' = \frac{2r+e+2}{m} \).

(a) We first consider the case \( m \geq 3 \):

(i) If \( d' \geq r + 2 \), there is a \( r \)-secant-(\( r - 2 \))-plane to \( C' \) which induces a \( g^1_{d'-r} \) on \( C' \) hence a \( g^1_{m(d'-r)} \) on \( C \). Then \( m(d'-r) - 2 = e - (m-2)r < e \), a contradiction.

(ii) In case \( m \geq 3 \) and \( d' = r \), \( C \) has a \( g^1_m \) and \( m - 2 = \frac{e+2}{r} \geq e \), which is only possible for \( m = 3, r = 3 \) and \( e = 1 \); in other words, \( C \) is trigonal.

(iii) In case \( m \geq 3 \) and \( d' = r + 1 \) with \( C' \) an elliptic curve, \( C \) has a \( g^1_{2m} \) and hence \( 2m - 2 = 2 + 2e + 2 - 2 \geq e \), which is only possible for \( e = 4, m = 3 \) and \( r = 3 \); i.e. \( C \) is a triple cover of an elliptic curve.

(b) For the case \( m = 2 \), \( C' \) is birational to a curve \( C'' \) in \( \mathbf{P}^3 \) of degree \( \frac{r}{2} + 4 \) with a complete \( g^3_{\frac{r}{2}+4} \) by projecting from \( (r - 3) \)-general points on \( C' \) in \( \mathbf{P}^r \). We then note the fact that \( C'' \) cannot have a 4-secant line; if there were a
4-secant line on $C''$, there would be a $g^1_3$ on $C''$ hence a $g^1_e$ on $C$, which is a contradiction. By the same computation which was carried out in [M] (lemma 2), we deduce that $(e, g') = (e, g(C'')) = (2, 2), (4, 4), (2, 1), (4, 3), (6, 5)$ or $(10, 10)$. But the case $(e, g') = (2, 1)$ does not occur; if this were the case, then $g^r_{2r+4}$ on $C$ is the pull-back of a $g^r_{r+2}$ on $C'$ which is not complete on an elliptic curve $C'$. On the base curve $C$ without base point on a curve $C$, the induced linear series $g^r_d$ gives rise to a $g^1_4$ on $C''$ which is not complete on an elliptic curve $C'$. We generalize (2.2) and (2.3) in the following theorem.

**Theorem 2.5.** Let $|D| = g^r_{2r+e+k}$, $r \geq 3$, $k \geq 0$ be a special linear series without base point on a curve $C$ with Clifford index $e$ such that $r(K - D) \geq 1$. If $r \geq 2k + 3$ then $|D|$ is either birationally very ample or $2:1$ to a curve of genus $e - k$; the last possibility does not occur if $e \geq k + 3$.

**Proof.** Assume that $|D|$ is birationally very ample. Then $|D|$ defines a morphism $C \rightarrow \mathbf{P}^{r'}$ of degree $m \geq 2$ onto a curve $C'$ in $\mathbf{P}^{r'}$ of degree $d' = \frac{2r+e+k}{m}$ and the induced linear series $g^r_{d'}$ on $C''$ gives rise to a $g^1_{d'-(r-1)}$ by taking off $(r-1)$-general points. Then this in turn induces a $g^1_{2r+e+k-m(r-1)}$ on $C$ whose Clifford index is at most $2r+e-k-m(r-1)-2 = (2-m)(r-1)+e+k$. But if $m \geq 3$, $(2-m)(r-1)+e+k < e$, which is a contradiction.

For the case $m = 2$, we consider the following two cases.

(i) The complete hyperplane series $|D'| = g^r_{r+e+k}$ on $C'$ is special: In this case we take off $(r-1)$-general points of $C'$ from the hyperplane series $|D'|$ and then pull it back to $C$, to get at least a $(r-1)$-dimensional family of $g^1_{2+e+k}$'s (possibly incomplete) on $C$. Thus for some $a \geq 1$, there exists at least a $(r-1) - 2(a - 1) = (r-1) - \dim \mathbf{G}(1,a)$ dimensional family of complete $g^a_{2+e+k}$'s on $C$. In other words, we have

$$\dim W^a_{2+e+k}(C) \geq (r-1) - 2(a - 1)$$

and hence

$$\dim W^1_{2+e+k-(a-1)}(C) \geq (r-1) - 2(a - 1) + (a - 1) = r - a.$$
On the other hand, by applying the basic inequality about the excess linear series (see [FHL]) to the above inequality and by hypothesis \( r \geq 2k + 3 \), \( \dim W^1_{e+1}(C) \geq (r - a) - 2(k - a + 2) \geq 0 \) which is a contradiction.

(ii) If \(|D'|\) on \(C'\) is non-special, then \(g' = \text{genus of } C' = \frac{e + k}{2}\). By the existence of a pencil of degree

\[
d'' \leq \left[ \frac{g' + 3}{2} \right] \leq \frac{e + k + 6}{4}
\]

on \(C'\), there exists a pencil of degree at most \(\frac{e + k + 6}{2}\) on \(C\), hence \(\frac{e + k + 6}{2} - 2 \geq e\).

### 3. Higher order Clifford indices

Given a curve \(C\) with Clifford index \(e\), set \(e_1 := e\). For each \(j \in \mathbb{N}\), define \(T(j) = \{|D| : 0 < \deg(D) < 2g - 2, |D| \text{ and } |K - D| \text{ are base-point-free and } \text{Cliff}(D) = j\} \). Let \(J = \{j \in \mathbb{N} : T(j) \neq \emptyset\} \). By definition of the Clifford index, \(J \neq \emptyset\) and \(e = \min(J)\). For each \(j \in J\), we call \(|D| \in T(j)\) an admissible linear series. Set \(e_2 := \min(J\setminus\{e\})\) if \(J \neq \{e\}\).

In general if we have defined \(e_1, \ldots, e_k\) and \(J \neq \{e_1, \ldots, e_k\}\), let \(e_{k+1} = \min(J\setminus\{e_1, \ldots, e_k\})\). These integers are called higher order Clifford indices of \(C\), \(e_k\) being the \(k\)-th Clifford index of \(C\). We call \(J\) the Clifford set of \(C\) and its cardinality is naturally an invariant of \(C\).

**Remark 3.1.** (i) For each \(e_j \in J\), let \(r_j = \min\{|D| : |D| = g_{d_j}^{r_j} \in T(e_j)\}\). If \(g_{d_j}^{r_j}\) is birational for some \(j \geq 2\) with \(r_j \geq 3\), this condition gives rather strong restrictions on the corresponding map; e.g. every \((2s + 2)\)-secant \(s\)-plane \((0 \leq s \leq r_j - 2)\) must be at least a \(2s + 2 + (e_j - e_{j-1})\)-secant.

(ii) If \(e_2 = e + 1\), then every \(|D| \in T(e_2)\) is birationally very ample by Proposition (2.2).

**Example 3.2.** If \(C\) has second Clifford dimension at least 2, i.e. \(r_2 > 1\), then an admissible \(|D| = g_{2r_2+e_2}^{r_2}\) is birationally very ample unless \(C\) is a \(e_2 - e + 2\) sheeted cover of a smooth plane curve of degree \(\frac{e_2 + 4}{r_2 - e + 2}\) and \(r_2 = 2\). In particular, if \(C\) is a \(e_2 - e + 2\) sheeted cover of a smooth conic, then \(e_2 = 2e\) and \(|D|\) is a double of a pencil \(g_{e+2}^{1}\) which computes the Clifford index.

**Proof.** Let’s assume that the map \(h\) induced by \(|D|\) has degree \(k > 1\). Then \(\deg C' = \deg h(C) = \frac{e_2 + 2e}{k}\). Fix a general \(p \in C'\) and set \(E = h^*(p)\).
By construction \(|D - E|\) is a base-point-free \(g^{r_2-1}_{2e_2+e_3-k}\) and is admissible; we note that no point outside \(E\) can be a base point of \(|K - D + E|\) since \(|K - D|\) has no base point. Then for any \(Q \in \text{Supp}(E)\), \(h^0(D - E + Q) = h^0(D - E)\) by construction. Furthermore \(
abla(D - E) = e_2 - k + 2\) and by the minimality of \(r_2, k > 2\). Since \(
abla(D - E) < 
abla(D)\), we must have \(
abla(D - E) = e_2 - k + 2 = e\). We now consider the following three cases:

(i) Suppose that \(r_2 \geq 4\). Since \(|D - E|\) computes the Clifford index, \(|D - E|\) is birationally very ample by Theorem 1 of [KKM]. On the other hand \(|D - E|\) induces a map of degree at least \(k\) by our construction, whence a contradiction.

(ii) If \(r_2 = 3, |D - E| = g^2_{6+e_2-k} = g^2_{e+4}\). But this cannot happen either since it would induce a \(g^1_{e+4-k'} (k' \geq 3)\) on \(C\), which is a pull-back of a pencil \(g^1_{e+4-k'}\) on the image curve of the map given by \(|D - E|\).

(iii) If \(r_2 = 2\), \(C'\) must be smooth since otherwise \(C\) would have \(g^1_{e+4-tk}\) with \(t > 1\). In case \(\deg C' = 2\), we have \(e_2 = 2e\). Furthermore in this case \(|D|\) is a double of a pencil \(g^1_{e+2}\) on \(C\) which computes the Clifford index.

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