THE RIEMANN PROBLEM FOR A SYSTEM OF CONSERVATION LAWS OF MIXED TYPE (I)

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ABSTRACT. We prove the existence of solutions of the Riemann problem for a system of conservation laws of mixed type using the method of vanishing viscosity term.

0. Introduction

In this paper we study the existence of solutions of the Riemann Problem for a 2 × 2 system of conservation laws of the mixed type

\begin{align*}
    u_t - f(v)_x &= 0, \\
    v_t - g(u)_x &= 0
\end{align*}

with the initial data

\begin{align*}
    (u, v)(x, 0) = \begin{cases} 
    (u_+, v_+) & x > 0, \\
    (u_-, v_-) & x < 0.
    \end{cases}
\end{align*}

Here we assume

(I) \( f \in C^2(\mathbb{R}) \) is a strictly increasing convex function.

(II) \( g \in C^2(\mathbb{R}) \) and there exist \( \alpha, \beta, \eta \) with \( \alpha < \eta < \beta \) such that

\begin{align*}
    g'(u) &\geq 0 \text{ if } u \notin (\alpha, \beta) \text{ and } g'(u) < 0 \text{ for } u \in (\alpha, \beta), \\
    g''(u) &< 0 \text{ if } u < \eta \text{ and } g''(u) > 0 \text{ if } u > \eta.
\end{align*}

(III) \( g(u) \to \pm \infty \text{ as } u \to \pm \infty. \)

If \( f(v) = v \), then the typical model of this equation (0.1) is the one-dimensional isothermal motion of a compressible elastic fluid or solid in

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the Lagrangian coordinates. In this case the existence of solutions to the
Riemann problem (0.1), (0.2) has been studied by Dafermos[1], Dafermos
and DiPerna[2], Fan[3], James[4], Slemrod[6]. These approach was based
on a vanishing "viscosity" term pursued by Kalashnikov[5], Tupchiev[7][8].
Their idea is to replace (0.1) with the system
\[
\begin{align*}
    u_t - f(v)_x &= \epsilon t u_{xx}, \\
    v_t - g(u)_x &= \epsilon t v_{xx},
\end{align*}
\]
for \( x \in \mathbb{R}, \ t > 0 \) and construct solutions as the limit of the solutions of
(0.3), (0.2) as \( \epsilon \to 0^+ \). Since the system is invariant under the transfor-
mation \((x, t) \rightarrow (ax, at)\) where \( a > 0 \), (0.3) and (0.2) admit solutions of
the form \((u_\epsilon(\xi), v_\epsilon(\xi))\), where \( \xi = \frac{x}{t} \). A simple computation shows that
\( u = u_\epsilon(\xi), v = v_\epsilon(\xi) \) is a solution of (0.3), (0.2) if it satisfies
\[
\begin{align*}
    -\xi u' - f(v)' &= \epsilon u'', \\
    -\xi v' - g(u)' &= \epsilon v''
\end{align*}
\]
with the boundary condition
\[
(0.5) \quad (u, v)(\pm \infty) = (u_\pm, v_\pm)
\]
where \( ' = \frac{d}{d\xi} \) and \( '' = \frac{d^2}{d\xi^2} \). We shall call the boundary value problem (0.4)
and (0.5) the problem \((P_\epsilon)\). Similarly the initial value problem (0.1) and (0.2)
are called the Riemann problem \((P)\). This paper consists of two parts. The first
part carried out in Section 1 and 2 establishes that if the data are in different
phases there is solution of \( P_\epsilon \) which exhibits one change of phase. In order
to prove the results, we use the arguments of Dafermos[1] and Slemrod[6].
In second part in Section 3 and 4 we prove the existence of solution to the
Riemann problem to give conditions on which solutions of \( P_\epsilon \) possess limits.
Throughout this paper we always assume Assumptions (I) and (II) unless
other mentions it.

1. The existence theorem of the problem \((P_\epsilon)\)

In this section we will study the existence of solutions to the boundary
value problem
\[
\begin{align*}
    \epsilon u'' &= -\xi u' - \mu f(v)', \\
    \epsilon v'' &= -\xi v' - \mu g(u)', \\
    (u, v)(\pm L) &= (u_\pm, v_\pm)
\end{align*}
\]
where $L > 1$, and $0 \leq \mu \leq 1$.

**Theorem 1.1.** Assume $u_- < \alpha, u_+ > \beta$ and there exists a constant $M_0$ such that every possible solution of (1.1) with $u'(\xi) > 0$ when $\alpha \leq u(\xi) \leq \beta$ satisfies the a priori estimate

\[
\sup_{|\xi| < L} (|u(\xi)| + |u'(\xi)| + |v(\xi)| + |v'(\xi)|) \leq M_0
\]

then $P_\epsilon$ has a solution with $u'(\xi) > 0$ if $\alpha \leq u(\xi) \leq \beta$.

**Proof.** Let $u_- < \alpha, u_+ > \beta$. Set $U(\xi) = u(\xi) - u_0(\xi)$ and $V(\xi) = v(\xi) - v_0(\xi)$, where $(u_0(\xi), v_0(\xi))$ is a unique solution of (1.1) with $\mu = 0$.

Then $U(-L) = U(L) = V(L) = V(-L) = 0$. If $u$ and $v$ are solutions of (1.1), $U, V$ satisfy

\[
\epsilon U'' = -\xi U' - \mu f(V + v_0)',
\]
\[
\epsilon V'' = -\xi V' - \mu g(U + u_0)'.
\]

Define

\[
Y(\xi) = \begin{pmatrix} U(\xi) \\ V(\xi) \end{pmatrix}, F(\xi, Y) = \begin{pmatrix} -f(V + v_0) \\ -g(U + u_0) \end{pmatrix}.
\]

Then

\[
\epsilon Y'' = -\xi Y' - \mu F(\xi, Y)',
\]
\[
Y(-L) = Y(L) = 0.
\]

Let $Z \in C^1([-L, L]; \mathbb{R}^2)$. Define $T$ to be the solution map that carries $Z$ into $Y$ where $Y$ solves

\[
\epsilon Y'' = -\xi Y' + F(\xi, Z)',
\]
\[
Y(-L) = Y(L) = 0.
\]

The integral formula of (1.4) is of the form

\[
Y(\xi) = c \int_{-L}^{\xi} \exp \left(-\frac{\tau^2}{2\epsilon} \right) d\tau + \frac{1}{\epsilon} \int_{-L}^{\xi} F(\tau, Z(\tau)) d\tau
\]
\[
+ \frac{1}{\epsilon^2} \int_{-L}^{\xi} \int_{0}^{\xi} \tau F(\tau, Z(\tau)) \exp \left(-\frac{\tau^2 - \xi^2}{2\epsilon} \right) d\tau d\xi.
\]
where

\[ c \int_{-L}^{L} \exp \left( -\frac{\xi^2}{2\epsilon} \right) d\xi = -\frac{1}{\epsilon} \int_{-L}^{L} F(\xi, Z(\xi)) d\xi \]

\[ + \frac{1}{\epsilon^2} \int_{-L}^{L} \int_{0}^{\xi} \tau F(\tau, Z(\tau)) \exp \left( \frac{\tau^2 - \xi^2}{2\epsilon} \right) d\tau d\xi \]

Then \( T : C^1([L, L]; \mathbb{R}^2) \rightarrow C^1([-L, L]; \mathbb{R}^2) \) is continuous and compact.

Define \( \Omega_1 \) by the set of pairs \( U, V \) in \( C^1([-L, L]; \mathbb{R}^2) \) such that

\[ U(-L) + u_0(-L) < \alpha, \quad U(L) + u_0(L) > \beta \]

\[ U'(\xi) + u'_0(\xi) > 0 \text{ if } \alpha \leq U(\xi) + u_0(\xi) \leq \beta \]

\[
\sup_{|\xi| < L} |U(\xi) + u_0(\xi)| + |U'(\xi) + u'_0(\xi)| + |V(\xi) + v_0(\xi)| + |V'(\xi) + v'_0(\xi)|
\]

\[ \leq M + 1 \]

Then \( \Omega \) is open and \( 0 \in \text{int} \Omega \).

We note that \( \phi \in \partial \Omega, \phi = \mu T \phi, \mu \in (0, 1) \) if and only if there is a solution \((u(\xi), v(\xi))\) of (1.1) satisfying \( u'(\xi) \geq 0 \) if \( \alpha \leq u(\xi) \leq \beta \) and either

(i) \( u'(\xi_0) = 0, \alpha \leq u(\xi_0) \leq \beta \) for some \( \xi_0 \in (-L, L) \)

or

(ii) \( \sup_{-L < \xi < L} |u(\xi)| + |v(\xi)| + |u'(\xi)| + |v'(\xi)| = M_0 + 1 \)

or both (i) and (ii).

The following lemma proved by Dafermos[1] is often useful.

**Lemma 1.2.** The initial value problem for (1.3), with fixed \( \epsilon > 0, \mu \in [0, 1] \), has a unique solution.

In order to use the Leray-Schauder fixed theorem, we take the Banach space \( X = C^1([-L, L]; \mathbb{R}^2) \).

Let us consider the case (i): either \( \alpha < u(\xi_0) < \beta, u(\xi_0) = \alpha, \) or \( u(\xi_0) = \beta \).

Case 1. \( \alpha < u(\xi_0) < \beta, u(\xi_0) = \alpha, u(\xi_0) = \beta \). Using Lemma 1.2 and the same method of Slemrod’s proof[6], we can not satisfy (1.1), \( u_- < \alpha, u_+ > \beta \).

Case 2. \( u(\xi_0) = \alpha, u'(\xi_0) = 0 \). In this case there are the three possibilities, \( u''(\xi_0) > 0, u''(\xi_0) = 0, \) or \( u''(\xi_0) < 0 \). The first and second cases are same
Thus \((\text{fixed point theorem, (1.1) possesses a solution for which (ii) cannot hold either. Thus from Leray-Schauder})\),

\[\alpha \leq |u|\] as Case 1. So we need only consider \(u''(\xi_0) < 0\). In this case \(u(\xi_0) = \alpha\) is a local maximum. Hence if \(u(L) = u_+ > \beta\), the local maximum of \(u\) occurs at \(\xi_1 > \xi_0\), i.e. \(u(\xi_1) < \alpha\), \(u'(\xi_1) = 0\), \(u''(\xi_1) \geq 0\); \(u(\xi) < \alpha\), \(u'(\xi) < 0\), \(\xi_0 < \xi \leq \xi_1\). The case \(u''(\xi_1) = 0\) is impossible because of \(v'(\xi_1) = 0\) and the Lemma 1.2. Thus we only consider \(u''(\xi_1) > 0\). From (1.1) and the assumption(I) of \(f\) we see that \(v(\xi_1) < 0\) and \(v(\xi_0) > 0\) which implies \(v\) has a local maximum at a point \(\xi_0 < \zeta < \xi_1\), \(u(\zeta) \leq 0\), and again Lemma 1.2 shows that \(v''(\zeta) > 0\). Since \(g'(u) > 0\) for \(u < \alpha\) this implies by use of (1.1) that \(u'(\zeta) > 0\) which contradicts the fact that \(u\) is decreasing on \((\xi_0, \xi_1)\).

Case 3. \(u(\xi_0) = \beta\), \(u'(\xi_0) = 0\). This case is similar to Case 1.

From Case 1, 2, 3 of (i) there is no solution of (1.1), \(\mu \in (0, 1)\), \((u(\xi) - u_0(\xi), v(\xi) - v_0(\xi))\) in \(\Omega\) for which (i) can hold. Thus all solutions of (1.1), \(\mu \in (0, 1)\) in \(\Omega\) must satisfy \(u'(\xi) > 0\) in \(\alpha \leq u(\xi) \leq \beta\). But the hypothesis of our theorem, (ii) cannot hold either. Thus from Leray-Schauder fixed point theorem, (1.1) possesses a solution for which \((u(\xi) - u_0(\xi), v(\xi) - v_0(\xi))\) is in \(\Omega\). To extend the domain of \(u, v\) as follows: Set

\[u(\xi; L) = u_+, v(\xi; L) = v_+ \text{ if } \xi > L,\]
\[u(\xi; L) = u_-, v(\xi; L) = v_- \text{ if } \xi < -L.\]

The extended pair \((u(\cdot; L), v(\cdot; L))\) form a sequence in \(C^0((-\infty, \infty); \mathbb{R}^2)\) and by virtue of the hypothesis of theorem we know \(\sup_{|\xi| < L} |u'(\xi; L)| + |v'(\xi; L)| \leq M\). Thus the sequence \(\{(u(\xi; L), v(\xi; L))\}\) is precompact in \(C^0((-\infty, \infty); \mathbb{R}^2)\) and so there is a subsequence \(L_n \rightarrow \infty\) as \(n \rightarrow \infty\) since that \((u(\xi; L), v(\xi; L)) \rightarrow (u(\xi), v(\xi))\) uniformly as \(n \rightarrow \infty\) on \((-\infty, \infty)\). Thus \((u(\xi), v(\xi))\) is a solution of \(P_\epsilon\) and by its construction \(u'(\xi) \geq 0\) if \(\alpha \leq u(\xi) \leq \beta\). But by the same reason used in Cases 2 and 3 \(u'(\xi) > 0\) if \(\alpha \leq u(\xi) \leq \beta\). This completes the proof of Theorem 1.1.

**Remark 1.3.** The conclusion of Theorem 1.1 remains valid if (1.2) is replaced by the a priori estimate

\[\sup_{|\xi| < L} (|u(\xi)| + |v(\xi)|) \leq M_1\]

where \(M_1 = M_1(u_-, v_- , u_+, v_+, \epsilon, f, g)\) but is independent of \(\mu\) and \(L\).
Remark 1.4. Assume $v_- > v_+$ and $u_-, u_+ < \alpha (v_- < v_+ \text{ and } u_-, u_+ > \beta)$ and there exist a constant $M_2$ such that every possible solution of (1.1) satisfies the a priori estimate

$$\sup_{|\xi| < L} |u(\xi)| + |v(\xi)| \leq M_2$$

Here $M_2 = M_2(v_-, v_+, u_-, u_+, \epsilon, f, g)$ but not independent of $\mu$ and $L$. Then there exist solutions of $(P_\epsilon)$ which satisfy the constraints $u(\xi) < \alpha$ and $u(\xi) > \beta$.

2. The a priori estimates

In this section we derive the a priori estimates needed to apply Theorem 1.1 and Remark 1.3 and 1.4. We give a series of Lemmas which is useful. Lemma 2.1 is a result of Dafermos[1].

Lemma 2.1. Let $(u(\xi), v(\xi))$ be a solution of (1.1) on $[-L, L]$, $\mu > 0$. Then on any subinterval $(l_1, l_2)$ for which $g'(u(\xi)) > 0$ one of the following holds:

(i) $u(\xi)$ and $v(\xi)$ are constant on $(l_1, l_2)$.

(ii) $v(\xi)$ is a strictly increasing (or decreasing) function with no critical points in $(l_1, l_2)$; $u(\xi)$ has, at most, one critical point in $(l_1, l_2)$ that necessarily must be a maximum (or minimum).

(iii) $u(\xi)$ is a strictly increasing (or decreasing) function with no critical point in $(l_1, l_2)$; $v(\xi)$ has, at most, one critical point in $(l_1, l_2)$ that necessarily must be a maximum (or minimum).

Lemma 2.2. $(u(\xi), v(\xi))$ be a solution of (1.1) on $[-L, L]$, $\mu > 0$. Then on any subinterval $(l_1, l_2)$ for which $g'(u(\xi)) < 0$ the graph of $v = v(u)$ is convex (or concave) at points where $u'(\xi) > 0$ (or $u'(\xi) < 0$).

Proof. Denote by $\frac{dv}{du} = \frac{v'(\xi)}{u'(\xi)}$. Then

$$\epsilon \frac{d^2v}{du^2} = \frac{\mu}{u'}(f'(v)(\frac{dv}{du})^2 - g'(u)).$$

The result follows from the above identity.

Lemma 2.3. $(u(\xi), v(\xi))$ be a solution of (1.1) on $[-L, L]$, $\mu > 0$ with $u'(\xi) > 0$ if $\alpha \leq u(\xi) \leq \beta$. Then $u$ and $v$ can have no local maxima or minima at $\xi$ for which $u(\xi) = \alpha$ or $u(\xi) = \beta$.
Proof. Since \( u'(\xi) > 0 \) if \( \alpha \leq u(\xi) \leq \beta \), \( u \) has no local maxima or a local minima at points where \( u(\xi) = \alpha \). On the other hand if \( v(\xi) \) has a local maximum or minimum at such a point, then \( v'(\xi) = 0 \) there and hence by (1.1) \( v''(\xi) = 0 \) as well. Differentiating (1.1) with respect to \( \xi \), \( g''(\alpha) < 0 \), \( g''(\beta) > 0 \) implies that \( u''(\xi) = 0 \) at such points, so \( u \) could not have taken on a local maximum or minimum.

Lemma 2.4 is the same result as Slemrod[6]. The proof is similar to his Lemma 2.4.

Lemma 2.4. Assume that \( u_- < \alpha, u_+ > \beta \) and let \( u(\xi), v(\xi) \) be a solution of (1.1) with \( \mu > 0 \) for which \( u'(\xi) > 0 \) when \( \alpha \leq u(\xi) \leq \beta \). Then one of the following holds: (0) No extreme points: \( u(\xi), v(\xi) \) have no local maxima or minima on \([-L, L]\). They are non-constant and monotone, \( u \) being monotone increasing.

(i) One extreme point: \( (a) \) \( u(\xi) \) has a minimum at some \( \xi_-, u(\xi_-) < u_-; v(\xi) \) is decreasing on \([-L, L]\). \( (b) \) \( u(\xi) \) has a maximum at some \( \xi_+, u(\xi_+) > u_+; v(\xi) \) is decreasing on \([-L, L]\). \( (c) \) \( v(\xi) \) has a maximum at some \( \eta_- \) (or \( \eta_+ \)); \( u(\eta_-) < \alpha \) (or \( u(\eta_+) > \beta \)) and \( u(\xi) \) is increasing on \([-L, L]\). \( (d) \) \( v(\xi) \) has a minimum at some \( \eta; \alpha < u(\eta) < \beta \) and \( u(\xi) \) is increasing on \([-L, L]\).

(ii) Two extreme points: \( (a) \) \( v(\xi) \) has a local maximum at \( \eta_- \) (or \( \eta_+ \)) and a local minimum at \( \eta \), \( u(\xi) \) is increasing on \([-L, L]\) and \( u_- < u(\eta_-) < \alpha \) (or \( u_+ > u(\eta_+) > \beta \)), \( \alpha < u(\eta) < \beta \). \( (b) \) \( u(\xi) \) has a minimum at \( \xi_- \), \( u(\xi_-) < u_-; v(\xi) \) has a local minimum at \( \eta \), \( \eta > \xi_- \), \( \alpha < u(\eta) < \beta \). \( (c) \) \( u(\xi) \) has a maximum at \( \xi_+, u(\xi_+) > u_+; v(\xi) \) has a local minimum at \( \eta \), \( \eta < \xi_+, \alpha < u(\eta) < \beta \).

(iii) Three extreme points: \( (a) \) \( v(\xi) \) has local maxima at \( \eta_- \), \( \eta_+ \) and a local minimum at \( \eta \), \( \eta_- < \eta < \eta_+ \); \( u(\xi) \) is increasing with \( u_- < u(\eta_-) < \alpha \), \( \alpha < u(\eta) < \beta \), \( \beta < u(\eta_+) < u_+ \). \( (b) \) \( u(\xi) \) has a minimum at \( \xi_- \), \( u(\xi_-) < u_- \) and maximum at \( \xi_+, u(\xi_+) > u_+ \) and \( v(\xi) \) has a local minimum at \( \eta \), \( \xi_- < \eta < \xi_+ \), \( \alpha < u(\eta) < \beta \). \( (c) \) \( u(\xi) \) has a minimum at \( \xi_- \), \( u(\xi_-) < u_- \), \( v(\xi) \) has a local minimum at \( \eta \), \( \alpha < u(\eta) < \beta \) and a local maximum at \( \eta_+ \), \( \eta < u(\eta_+) < u_+ \), \( \xi_+ < \eta < \eta_+ \). \( (d) \) \( u(\xi) \) has a maximum at \( \xi_+, u(\xi_+) > u_+ \), \( v(\xi) \) has a local maximum at \( \eta_- \), \( u_- < u(\eta_-) < \alpha \), and a local minimum at \( \eta \), \( \alpha < u(\eta) < \beta \).

Theorem 2.5. Assume \( u_- < \alpha, u_+ > \beta \) \((u_- > \beta, u_+ < \alpha)\). Then there exist constant \( M_1 \) such that every possible solution of (1.1), \( 0 \leq \mu \leq 1 \), with
$u'(\xi) > 0$ ($u'(\xi) < 0$) when $\alpha \leq u(\xi) \leq \beta$ satisfies
\[
\sup_{|\xi| < L} (|u(\xi)| + |v(\xi)|) \leq M_1
\]
where $M_1$ depends at most on $u_-, u_+, v_-, v_+, \epsilon, f, g$ and is independent of $\mu$ and $L$.

Proof. We will prove the case $u_- < \alpha, u_+ > \beta$. The proof for $u_- > \beta, u_+ < \alpha$ is similar.

The case (0) is nothing to prove.

The case (ia) Since v is decreasing, $v_+ \leq v(\xi) \leq v_-$. Since $u$ has a minimum at $\xi_-$, we need only bound $u$ from below. Assume $\xi_- \geq 0$. In case $\xi_- \leq 0$ will be similarly proved. Integrating (1.1) from $\xi_-$ to $L$ and use $u'(\xi_-) = 0$, we have
\[
\epsilon u'(L) + \int_{\xi_-}^{L} \xi u'(\xi) d\xi \leq -\mu f(v_+) + \mu f(v(\xi_-)).
\]
Since $u'(L) > 0$, we have
\[
\int_{\xi_-}^{L} \xi u'(\xi) d\xi \leq -\mu f(v_+) + \mu f(v(\xi_-)).
\]
If $\xi \geq \max\{1, \xi_-\}$, then $u'(\xi) \leq \xi u'(\xi)$ on $(\xi, L)$ so that
\[
u(L) - u(\xi) \leq -\mu f(v_+) + \mu f(v(\xi_-)).
\]
and hence
(2.1) \[ u(\xi) \geq u_+ + \mu f(v_+) - \mu f(v(\xi_-)). \]
Since $v_+ \leq v(\xi_-) \leq v_-, 0 \leq \mu \leq 1$, we have
\[ u(\xi) \geq u_+ + f(v_+) - f(v_-) \] if $\xi_- \geq 1$.
If $0 \leq \xi_- < 1$, integrate (1.1) from $\xi_-$ to $\xi$ where $\xi_- < \xi < 1$, then
\[
\epsilon u'(\xi) + \int_{\xi_-}^{\xi} \xi u'(\xi) d\xi = -\mu f(v(\xi)) + \mu f(v(\xi_-)).
\]
Since \( u'(\xi) > 0 \) on \( (\xi_-, L) \), we obtain \( \zeta u'(\zeta) > 0 \) and
\[
(2.2) \quad \epsilon u'(\xi) \leq -\mu f(v(\xi)) + \mu f(v(\xi_-)), \quad \xi_- < \xi < 1.
\]
Integrate (2.2) from \( \xi_- \) to 1, We see that
\[
(2.3) \quad \epsilon u(1) - \epsilon u'(\xi_-) \leq -\mu \int_{\xi_-}^{1} (f(v(\xi)) + \mu f(v(\xi_-))) \, d\xi.
\]
Since \( v_+ \leq v(\xi) \leq v_- \) and \( u(1) \) is bounded from below by (2.1), (2.3) implies
that \( u(\xi_-) \) is bounded from below when \( 0 \leq \xi_- < 1 \).

The cases (ib) and (ic) are proven similarly.

The case (id): Since \( u(\xi) \) is increasing so \( u_- \leq u(\xi) \leq u_+ \). Assume that \( \eta \geq 0 \). In case \( \eta < 0 \) is similar. First integrate (1.1) from \( \eta \) to \( L \), this implies
\[
\epsilon v'(L) + \int_{\eta}^{L} \xi u'(\xi) \, d\xi = -\mu g(u_+) + \mu g(u(\eta)).
\]
Since \( v'(L) > 0 \) this implies
\[
\int_{\eta}^{L} \xi u'(\xi) \, d\xi \leq -\mu g(u_+) + \mu g(u(\eta)).
\]
If \( \zeta \geq \max\{1, \eta\} \), since \( v'(\xi) > 0 \) on \( (\zeta, L) \) we find \( v'(\xi) \leq \xi v'(\xi) \) on \( (\eta, L) \) and
\[
v_+ - v(\zeta) = \int_{\zeta}^{L} v'(\xi) \, d\xi \leq \int_{\eta}^{L} \xi u'(\xi) \, d\xi \leq -\mu g(u_+) + \mu g(u(\eta))
\]
Thus we have
\[
(2.4) \quad v(\zeta) \geq v_+ + \mu g(u_+) - \mu g(u(\eta)).
\]
Since \( \alpha < u(\eta) < \beta \), we see for \( \eta \geq 1 \)
\[
v(\eta) \geq v_+ + \mu g(u_+) - \mu g(\alpha) \geq v_+ - g(\alpha).
\]
Again if \( 0 \leq \eta < 1 \), integrate (1.1) from \( \eta \) to \( \xi \) where \( \eta < \xi < 1 \). Then we have
\[
\epsilon v'(\xi) + \int_{\eta}^{\xi} \xi u'(\xi) \, d\xi = -\mu g(u(\xi)) + \mu g(u(\eta)).
\]
Since \( \zeta v'(\xi) > 0 \) on \((\eta, \xi)\), we find

\[
e v'(\xi) \leq -\mu g(u(\xi)) + \mu g(u(\eta)).
\]

and integrate it from \(\eta\) to 1 we have

\[
e v(1) - e v(\eta) \leq -\mu \int_{\eta}^{1} (g(u(\xi)) - g(u(\eta))) \, d\xi.
\]

and

\[
(2.5) \quad e v(1) + \mu \int_{\eta}^{1} (g(u(\xi)) - g(u(\eta))) \, d\xi \leq e v(\eta).
\]

We know \(\max(v_-, v_+) \geq v(\xi)\) and so \(v\) is bounded from above. Since \(u(\xi)\) is bounded, (2.4) and (2.5) imply that \(v(\xi)\) is bounded from below on \([-L, L]\) independently of \(\mu\) and \(L\).

The case (iia): Assume \(v\) has a local maximum at \(\eta_-\), \(u(\eta_-) < \alpha\). The case \(u(\eta_+)\) is similar. Then the local minimum is at \(\eta, \eta_- < \eta, \alpha < u(\eta) < \beta\). For \(u\) we know \(u_- \leq u(\xi) \leq u_+\). In case there are two cases \(\eta \geq 0\) and \(\eta < 0\). If \(\eta \geq 0\), the same method of the proof of (id) implies the boundedness of \(v\). If \(\eta < 0\), then \(\eta_- < 0\). We will show \(u(\eta_-)\) is bounded from below. We consider first \(\eta_- \leq -1\) and then \(-1 \leq \eta_- \leq 0\). In the first case we use (ic) on \([-L, \xi \eta\) to bound \(v(\eta_-)\) from above; in the second case we use (id) on \(\eta_- \leq \xi \leq L\) to bound \(u(\xi)\) from below. These bounds is independent of \(\mu\) and \(L\).

The case (iib): If \(\eta \geq 0\), the argument of (id) says that \(v(\eta)\) is bounded from below. Since \(v(\eta)\) is bounded from above by \(\max(u_-, u_+)\), \(v(\xi)\) is bounded from above and below. Use (ia) on \([-\xi, \eta\), \(u\) is bounded from below at \(\xi_- \in (-L, \eta)\). If \(\eta < 0\), then argument of (id) implies

\[
v(\xi) \geq v_- + \mu g(u_-) - \mu g(u(\eta))
\]

if \(\xi \leq \min\{-1, \eta\}\). But \(\alpha < u(\eta) < \beta\) so \(u(\eta)\) is bounded from below if \(\eta \leq -1\). If \(-1 < \eta \leq 0\), argument (id) can be used again. First integrate (1.1) from \(\eta\) to \(\xi\) where \(\xi \in (-1, \eta)\). This implies

\[
e v'(\xi) + \int_{\eta}^{\xi} \xi v'(\xi) \, d\xi = \mu g(u(\eta)) - \mu g(u(\xi)).
\]
On \((\xi, \eta), \zeta v'(\xi) > 0\) so

\[\epsilon v'(\xi) \geq \mu g(u(\eta)) - \mu g(u(\xi)).\]

Now integrate (2.6) from -1 to \(\eta\),

\[\epsilon v(\eta) \geq \epsilon v(-1) + \mu \int_{-1}^{\eta} (g(u(\eta)) - g(u(\xi))) \, d\xi.\]

Now \(u(\xi) \leq u(\eta)\) on \((-1, \eta)\) since \(\alpha < u(\eta) < \beta\),

\[g(u(\eta)) - g(u(\xi)) \geq g(\beta) - g(\alpha).\]

Insert (2.8) into (2.7) we have

\[\epsilon v(\eta) \geq \epsilon v(-1) + \mu (\eta + 1)(g(\beta) - g(\alpha))\]

and hence

\[\epsilon v(\eta) \geq \epsilon v(-1) + \mu (g(\beta) - g(\alpha)).\]

Thus \(v(\eta)\) is bounded if \(\eta \leq 0\). Now use (ia) on \((-L, \eta)\) \(u(\xi)\) is bounded from below.

The case ii(c): This case is proved the same method of ii(b).

The case iii(a): Since \(u\) is monotone increase, \(u_- \leq u(\xi) \leq u_+\) on \([-L, L]\). As to \(v\), either \(\eta_+ \geq 0\) or not. If \(\eta \geq 0\), using the method of (ic) \(v(\eta_+)\) is bounded from above. If \(\eta_+ < 0\), then \(\eta_- < 0\) and again using the same method of (ic) \(v(\eta_-)\) is bounded from above. Thus if \(\eta_+ \geq 0\), \(u_+ \leq u(\eta_+) \leq M_1\); if \(\eta_+ < 0\) then \(u_- \leq u(\eta_-) \leq M_2\). This case is reduced to the case (iia).

The case iii(b): If \(\eta \geq 0\), then ii(c) implies that for \(\eta \geq 1\)

\[v(\eta) \geq v_+ - \mu g(u(\eta)) + \mu g(u_+).\]

Since \(\alpha \leq u(\eta) \leq \beta\), (2.9) shows that \(u(\eta)\) is bounded from below. If \(0 \leq \eta < 1\), ii(c) shows

\[\epsilon v(\eta) \geq \epsilon v(1) + \mu (g(\beta) - g(\alpha))\]

Thus \(v(\eta)\) is bounded from below. If \(\eta < 0\), ii(a) show \(u(\eta)\) is bounded from below. Thus \(v(\eta)\) is bounded from above and below.
The case iii(c) : If $\eta \leq 0$, then the proof is same as the method of ii(b). If $\zeta \leq \min\{-1, \eta\}$, then
$$v(\zeta) \geq v_- - \mu(g(u_-) - g(u(\eta))).$$
Since $\alpha \leq u(\eta) < \beta$, $v(\eta)$ is bounded from below if $\eta \leq -1$. If $-1 < \eta \leq 0$ we have
$$\epsilon v(\eta) \geq \epsilon v(-1) + \mu \int_{-1}^{\eta} g(u(\eta)) - g(u(\xi)) \, d\xi$$
where $u(\xi) \leq u(\eta), -1 \leq \xi \eta$. In this case
$$g(u(\eta)) - g(u(\xi)) \geq g(\beta) - g(\alpha)$$
and so
$$\epsilon v(\eta) \geq \epsilon v(-1) + \mu(g(\beta) - g(\alpha))$$
and $u(\eta)$ is bounded from above for $\eta \leq 0$. If $\eta \geq 0$, then $\eta _+ \geq 0$. The same argument of i(c) yields $v(\eta _+)$ is bounded from above. If $\zeta \geq \max\{\eta _+, 1\}$, we find
$$v(\zeta) \leq v_+ + \mu g(u_+) - \mu g(u(\eta _+)).$$
Since $\beta \leq u(\xi) \leq u_+$ for $\xi \in [\eta _+, 1]$, $v(\eta _+)$ is bounded from above if $\eta _+ \geq 1$. If $0 \leq \eta _+ < 1$, we find
$$\epsilon v(\eta _+) \leq \epsilon v(-1) + \mu \int_{\eta _+}^{-1} g(u(\xi)) - g(u(\eta _+)) \, d\xi.$$ 
But $\beta \leq u(\xi) \leq u_+$ for $\xi \in [\eta _+, 1]$, $v(\eta _+)$ is bounded from above. Then $\eta \leq 0$, $v(\eta)$ is bounded from above and below; if $\eta > 0$, then $v(\eta _+)$ is bounded from above and below.

The case iii(d) : The proof is similar of the proof of iii(c).

**Theorem 2.6.** Assume $v_+ < v_-$ and $u_+, u_+ < \alpha$ (or $v_- < v_+$ and $u_-, u_+ > \beta$). Then there is a constant $M_2$ such that every possible solution of (1.1), $0 \leq \mu \leq 1$, satisfies the a priori estimate
$$\sup_{|\xi| < L} (|v(\xi)| + |u(\xi)|) \leq M_2$$
where $M_2$ depends at most on $u_-, u_+, v_-, v_+, \epsilon, f, g$ and is independent of $\mu$ and $L$.

**Corollary 2.7.** If $u_- < \alpha, u_+ > \beta$ (or $u_- > \beta, u_+ < \alpha$), there are solutions of $(P_\epsilon)$ which satisfy the constants $u'(\xi) > 0, u'(\xi) < 0$ when $\alpha \leq u(\xi) \leq \beta$. If $v_+ < v_-$ and $u_-, u_+ < \alpha$ (or $v_- < v_+$ and $u_-, u_+ > \beta$) there are solutions of $P_\epsilon$ which satisfy the constraints $u(\xi) < \alpha(u(\xi) > \beta)$. 
3. Existence of Solutions of the Riemann problems assuming \( \{(u_\epsilon, v_\epsilon)\} \) are uniformly bounded.

In this section we prove the existence of solutions to the Riemann problem assuming the set \( \{(u_\epsilon, v_\epsilon)\} \) are uniformly bounded. Proposition 3.1 is a result of Dafermos[1].

**Proposition 3.1.** For fixed \( \epsilon > 0 \), let \( (u_\epsilon, v_\epsilon) \) denote a solution of \( P_\epsilon \). Suppose that the set \( \{(u_\epsilon, v_\epsilon) : 0 < \epsilon < 1\} \) is of uniformly bounded variation. Then \( \{(u_\epsilon, v_\epsilon)\} \) possesses a subsequence which converges almost everywhere on \((-\infty, \infty)\) of bounded variation. The pair \( u(\frac{\xi}{\epsilon}), v(\frac{\xi}{\epsilon}) \) provided a weak solution of \( P \).

Using Proposition 3.1, we have an existence theorem for the one phase case.

**Theorem 3.2.** If \( v_- > v_+ \) and \( u_- < \alpha(\text{or } u_+, u_+ > \beta) \) and Assumption (III) holds, the sequence \( \{(u_\epsilon(\xi), v_\epsilon(\xi)) ; 0 < \epsilon < 1\} \) as given by Corollary 2.7 possesses a subsequence which converges a.e. on \((-\infty, \infty)\) to function \( (u(\xi), v(\xi)) \) of bounded variation. The pair \( u(\frac{\xi}{\epsilon}), v(\frac{\xi}{\epsilon}) \) provides a solution to the Riemann problem \( P \) with \( u(\frac{\xi}{\epsilon}) < \alpha(\text{or } u(\frac{\xi}{\epsilon}) > \beta) \).

**Lemma 3.3.** The list for \( (u_\epsilon(\xi), v_\epsilon(\xi)) \) given in Lemma 2.4 is valid when \( L = \infty \).

**Lemma 3.4.** In case 0, i(a, b, c) of Lemma 2.4 \( (u_\epsilon(\xi), v_\epsilon(\xi)) \) are uniformly bounded independent of \( \epsilon \) on \((-\infty, \infty)\). That is, there is a constant \( N \) dependent on \( u_-, u_+, v_-, v_+, f, g \) and independent of \( \epsilon, 0 < \epsilon < 1 \) such that

\[
\sup_{|\xi| < \infty} (|u_\epsilon(\xi)| + |v_\epsilon(\xi)|) \leq N.
\]

**Proof.** Case 0: it is obvious. Case i(a): Since \( v_\epsilon(\xi) \) is monotone decreasing, \( v_+ \leq v_\epsilon(\xi) \leq v_- \) on \((-\infty, \infty)\). Denote \( \frac{du}{dv}(\xi) = \frac{u'(\xi)}{v'(\xi)} \). We claim that

\[
0 < \frac{du}{dv}(\xi) < \left( \frac{f'(v_\epsilon)}{g'(u_\epsilon)} \right)^{1/2} \quad \text{on} \quad (-\infty, \xi^-_1].
\]

Indeed, if not, set

\[
\xi_1 = \max \left\{ \xi \in (-\infty, \xi^-_1] : \frac{du}{dv}(\xi) \geq \left( \frac{f'(v_\epsilon)}{g'(u_\epsilon)} \right)^{1/2} \right\}.
\]
Since $u_\epsilon$ has its minimum at $\xi_\epsilon^-$, $\frac{du}{dv}(\xi_\epsilon^-) = 0$ and so $\xi_1 < \xi_\epsilon^-$ must exist. A simple computation shows that

$$\epsilon \frac{d}{d\xi} \left( \frac{du}{dv}(\xi) \right) = -f'(v_\epsilon) + g'(u_\epsilon) \left( \frac{du}{dv} \right)^2$$

and so $\epsilon \frac{d}{d\xi} \left( \frac{du}{dv}(\xi) \right) = 0$ at $\xi = \xi_1$. By the definition of $\xi_1$ we have

$$0 < \frac{du}{dv}(\xi) < \left( \frac{f'(v_\epsilon)}{g'(u_\epsilon)} \right)^{1/2} \quad \text{on} \quad (\xi_1, \xi_\epsilon^-)$$

and thus $\frac{d}{d\xi} \frac{du}{dv}(\xi) < 0$ on $(\xi_1, \xi_\epsilon^-)$ and $\frac{d^2}{d\xi^2} \frac{du}{dv}(\xi_1) < 0$. On the other hand, differentiation of (3.2) shows that

$$\epsilon \frac{d^2}{d\xi^2} \left( \frac{du}{dv}(\xi) \right) = -f''(v_\epsilon)v_\epsilon'(\xi) + g''(u_\epsilon)u_\epsilon'(\xi) \left( \frac{du}{dv} \right)^2 \quad \text{at} \quad \xi = \xi_1.$$

From Assumptions 1 and 2 it follows that

$$\frac{d^2}{d\xi^2} \left( \frac{du}{dv}(\xi) \right) > 0 \quad \text{at} \quad \xi = \xi_1.$$

This contradicts the assumption. Thus we see $\frac{d}{d\xi} \left( \frac{du}{dv}(\xi) \right) \leq 0$ on $(-\infty, \xi_\epsilon^-]$. Hence for any $\xi \in (-\infty, \xi_\epsilon^-]$,

$$\frac{du}{dv}(\xi) < \frac{du}{dv}(-\infty) = \left( \frac{f'(v)}{g'(u)} \right)^{1/2}.$$

Now

$$u_\epsilon(\xi^-) - u_- = \int_{v_-}^{v(\xi^-)} \frac{du}{dv} dv$$

$$> - \int_{v_-}^{v_-} \left( \frac{f'(v_-)}{g'(u_-)} \right)^{1/2} dv$$

$$= - \left( \frac{f'(v_-)}{g'(u_-)} \right)^{1/2} (v_- - v_\epsilon(\xi^-)),$$

which is bounded from below.
Case ii(b) : The proof is similar to ii(a).

Case ii(c) : Let $\eta^-_e$ be a point such that $v_e(\xi)$ has its maximum value and $u_e(\eta^-_e) < \alpha$. Since $u_e(\xi)$ is increasing, $u_- \leq u_e(\xi) \leq u_+$ on $(-\infty, \infty)$. Denote by $\frac{dv}{du}(\xi) = \frac{v'(\xi)}{u'(\xi)}$. We claim that $0 < \frac{dv}{du}(\xi) < (\frac{g'(u_e)}{f'(v_e)})^{1/2}$ on $(-\infty, \eta^-_e]$. For if not, set

$$\xi_1 = \max \left\{ \xi \in (-\infty, \eta^-_e] \mid \frac{dv}{du}(\xi) \geq \left( \frac{g'(u_e)}{f'(v_e)} \right)^{1/2} \right\}.$$

Since $\frac{dv}{du}(\xi) = 0$ at $\xi = \xi_1$, $\xi_1$ exist such that $\xi_1 < \eta^-_e$. A simple computation say

$$\epsilon \frac{d}{d\xi} \left( \frac{dv}{du}(\xi) \right) = -g'(u_e(\xi)) + f'(v_e(\xi)) \left( \frac{dv}{du}(\xi) \right)^{1/2}$$

implies $\frac{d}{d\xi} \left( \frac{dv}{du}(\xi_1) \right) = 0$. By the definition of $\xi_1$, $0 < \frac{dv}{du}(\xi) < (\frac{g'(u_e)}{f'(v_e)})^{1/2}$ on $(\xi, \eta^-_e]$. Thus we have $\frac{d}{d\xi} \left( \frac{dv}{du}(\xi) \right) < 0$ at $\xi = \xi_1$. On the other hand, differentiation of (3.3) gives

$$\epsilon \frac{d^2}{d\xi^2} \left( \frac{dv}{du}(\xi) \right) = -g''(u_e)u'_e(\xi) + f''(v_e)v'_e(\xi) \left( \frac{dv}{du}(\xi) \right)^2 > 0$$

at $\xi = \xi_1$, a contradiction. Thus we see that $\frac{d}{d\xi} \left( \frac{dv}{du}(\xi) \right) < 0$ on $(-\infty, \eta^-_e]$ and hence for any $\xi \in (-\infty, \eta^-_e]$,

$$0 < \frac{dv}{du}(\xi) < \frac{dv}{du}(-\infty) = \left( \frac{g'(u_-)}{f'(v_-)} \right)^{1/2}.$$

Then

$$v_e(\eta^-_e) - v_- = \int_{u_-}^{u_e(\eta^-_e)} \frac{dv}{du} du \leq \left( \frac{g'(u_-)}{f'(v_-)} \right)^{1/2} (u_e(\eta^-_e) - u_-).$$

Since $u_- \leq u(e\eta^-_e) \leq u_+$, we see that $u_e(\eta^-_e)$ is bounded from above, independent of $\epsilon$ for $u(\eta^-_e) < \alpha$. Analogous computation shows that if $u_e(\eta^+_e) > \beta$ we have

$$v_e(\eta^+_e) \leq v_+ + \left( \frac{g'(u_+)}{f'(v_+)} \right)^{1/2} (u_e(\eta^+_e) - u_-)$$

and since $u_- \leq u(\eta^+_e) \leq u_+$, a bound on $v_e(\eta^+_e)$ independent of $\epsilon$ is provided.
LEMMA 3.5. Let $\eta^\epsilon$ denote the points such that $v_\epsilon(\xi)$ takes on its local minimum, $\alpha < u_\epsilon(\eta^\epsilon) < \beta$. If there is a subsequence $\{\eta^{\epsilon_n}\}$ of $\{\eta^\epsilon\}$, $\epsilon_n \to 0+$ such that either (a) $\eta^{\epsilon_n} \geq m > 0$ or $\eta^{\epsilon_n} \leq -m < 0$, $m$ a constant independent of $\epsilon$, or (b) $v_\epsilon(\eta^{\epsilon_n})$ is bounded from below independently of $\epsilon$, then for Case i(d) $\{(u_\epsilon(\xi), v_\epsilon(\xi))\}$ satisfies (3.1).

Proof. Assume $\eta^{\epsilon_n} \leq m < 0$. Then $v_\epsilon'(\xi) \leq 0$ on $(-\infty, \eta^{\epsilon_n}]$ and $\xi v_\epsilon'(\xi) \geq -m v_\epsilon'(\xi)$ on $(-\infty, \eta^{\epsilon_n}]$. Now

$$-m(v_\epsilon(\eta^{\epsilon_n}) - v_-) \leq \int_{-\infty}^{-\frac{m}{v_\epsilon'(\xi)}} \eta^{\epsilon_n} \xi v_\epsilon'(\xi) d\xi$$

$$= \int_{-\infty}^{\eta^{\epsilon_n}} \eta^{\epsilon_n}(g'(u) - \epsilon_n v'') d\xi$$

$$= g(u(\eta^{\epsilon_n})) - g(u_-)$$

hence

$$\frac{1}{m}(g(u_-) - g(u(\eta^{\epsilon_n}))) + v_- \leq v(\eta^{\epsilon_n})$$

Since $u_\epsilon(\xi)$ is monotone, $u_- \leq u_\epsilon(\eta^{\epsilon_n}) \leq u_+$, we see that $v_\epsilon(\eta^{\epsilon_n})$ is bounded from below independently of $\epsilon$. The case $\eta^{\epsilon_n} \geq m > 0$ is similar. Thus in (a) or (b), $v(\eta^{\epsilon_n})$ is bounded for below and hence $\{(u_\epsilon(\xi), v_\epsilon(\xi)) | 0 < \epsilon < 1 \}$ satisfies (3.1).

LEMMA 3.6. In case ii(a,b,c), iii(a,b,c,d) assume $\{\eta^\epsilon\}$ satisfies the hypothesis of Lemma 3.4. Then $\{(u_\epsilon(\xi), v_\epsilon(\xi)) | 0 < \epsilon_n < 1 \}$ satisfies (3.1).

From Lemmas 3.4, 3.5, 3.6 and Prop 3.1 we have

THEOREM 3.7. Assume $u_- < \alpha, u_+ > \beta$ (or $u_- > \alpha, u_+ < \beta$) and let $(u_\epsilon(\xi), v_\epsilon(\xi))$ denote the solution of $P_\epsilon$ given by Corollary 2.7. Let Assumptions (II) and (III) and the hypothesis of Lemma 3.4 hold. Then $\{(u_\epsilon(\xi), v_\epsilon(\xi)) | 0 < \epsilon_n < 1 \}$ possesses a subsequence which converges almost everywhere on $(-\infty, \infty)$ to a function $(u(\xi), v(\xi))$ of bounded variation. The pair $(u(\xi), v(\xi))$ provides a solution of the Riemann problem.

REMARK 3.8. If the hypothesis of Lemma 3.5 does not hold then $\eta^\epsilon \to 0$, $v_\epsilon(\eta^\epsilon) \to -\infty$ as $\epsilon \to 0+$.
4. Existence of solutions to the Riemann problem: the case when \( v(\eta^e) \to -\infty \) as \( \eta^e \to 0 \).

In this section we will prove the existence of solution to the Riemann problem in case when \( v(\eta^e) \to -\infty \) as \( \eta^e \to 0 \). This situation was mentioned in Remark 3.8. First we must show that \( u^e(\xi), v^e(\xi) \) has a pointwise a.e. limit.

**Lemma 4.1.** Let \( (u^e(\xi), v^e(\xi)) \) be a solution of \( P^e \) as given by Corollary 2.7 when \( u_- < \alpha, u_+ > \beta \). Let \( \bar{v} = \min(v_-, v_+) \). Then if \( v^e(\xi) \) has a local minimum at \( \eta^e \) with \( \alpha < u^e(\eta^e) < \beta \), we have the estimate

\[
N_0(s_1 - s_2) \geq \int_{s_1}^{s_2} v^e(\xi) \, d\xi \geq \bar{v}(s_2 - s_1) + (g(\beta) - g(\alpha))
\]

(4.1)

\[
\bar{v} + \frac{g(\beta) - g(\alpha)}{|\xi - \eta^e|} \leq v^e(\xi) \leq N_0, \quad -\infty < \xi < \infty
\]

(4.2)

Here \( (s_1, s_2) \subset (-\infty, \infty) \) and \( N_0 \) is a constant independent of \( \epsilon \).

**Proof.** The bound from above on \( v^e(\xi) \) in (4.1), (4.2) follows from the proof of Lemma 3.3, 3.4, and 3.5. Thus we now proceed to get the bounds from below. i(d) Fix \( l < \infty \) sufficiently large so that \( u^e(-l) < \alpha, u^e(l) > \beta \). Assume for the moment \( v^e(-l) \leq v^e(l) \),and let \( \theta > -l \) be such that \( v^e(\theta) = v^e(-l) \). Then we have \( v^e(\xi) \leq v^e(-l) \) on \((-l, \theta)\), \( v^e(\xi) \geq v^e(-l) \) on \( \theta < \xi < \infty \) when \(-l < \eta^e < \theta < l \). From \( P^e \) we know that

\[
\epsilon(v^e(\xi) - v^e(-l))'' + \xi (u^e(\xi) - u^e(-l))' = -g(u^e)'
\]

(4.3)

and integration of (4.3) from \(-l\) to \( \theta \) shows that

\[
\epsilon(v^e(\theta) - v^e(-l)) - \int_{-l}^{\theta} (v^e(\xi) - v^e(-l)) \, d\xi = -g(u^e(\theta)) + g(u^e(-l))
\]

But \( v'(\theta) > 0, v'(-l) < 0 \) and hence

\[
\int_{-l}^{\theta} (v^e(-l) - v^e(\xi)) \, d\xi \leq g(u^e(-l)) - g(u^e(\theta))
\]

(4.4)
Since \( u_\epsilon(\theta) > u_\epsilon(-l) \), the right-hand side of (4.4) is bounded from above by \( g(\alpha) - g(\beta) \). Then for any \( (s_1, s_2) \subset (-l, \theta) \) we have

\[
(4.5) \quad \int_{s_1}^{s_2} (v_\epsilon(-l) - v_\epsilon(\xi)) \, d\xi \leq g(\alpha) - g(\beta)
\]

and hence

\[
v_\epsilon(-l)(s_2 - s_1) + (g(\beta) - g(\alpha)) \leq \int_{s_1}^{s_2} v_\epsilon(\xi) \, d\xi.
\]

Letting \( l \to -\infty \) we have

\[
(4.6) \quad \bar{v}(s_2 - s_1) + (g(\beta) - g(\alpha)) \leq \int_{s_1}^{s_2} v_\epsilon(\xi) \, d\xi.
\]

If \( (s_1, s_2) \subset (\theta, l) \), then \( v_\epsilon(\xi) \geq v_\epsilon(-l) \) and we see

\[
(4.7) \quad \bar{v}(s_2 - s_1) \leq \int_{s_1}^{s_2} v_\epsilon(\xi) \, d\xi.
\]

Finally if \( -l < s_1 < \theta, \theta < s_2 < l \), we write

\[
\int_{s_1}^{s_2} v_\epsilon(\xi) \, d\xi = \int_{s_1}^{\theta} v_\epsilon(\xi) \, d\xi + \int_{\theta}^{s_2} v_\epsilon(\xi) \, d\xi
\]

and use (4.6) and (4.7) to obtain (4.1) again. To get the bound from below in (4.2), we observe that when \( \eta^\epsilon < \xi < \theta \)

\[
(4.8) \quad (v_\epsilon(-l) - v_\epsilon(\xi))(\xi - \eta^\epsilon) \leq \int_{-l}^{\theta} (v_\epsilon(-l) - v_\epsilon(\xi)) \, d\xi.
\]

From (4.8) and (4.5) we see that

\[
(v_\epsilon(-l) - v_\epsilon(\xi))(\xi - \eta^\epsilon) \leq g(\alpha) - g(\beta)
\]

Now letting \( l \to \infty \) we obtain (4.2). If \( -l < \xi < \eta^\epsilon \) we again (4.2) and if \( \theta \leq \xi \leq l \), we also obtain (4.2). The proof for \( v_\epsilon(-l) > v_\epsilon(l) \) is analogous.
LEMMA 4.2. Let \( \{(u_\epsilon(\xi), v_\epsilon(\xi))|0 < \epsilon < 1\} \) be a solution of \((P_\epsilon)\) as given by Corollary 2.7 when \( u_- < \alpha, u_+ > \beta \). Then for any given compact subset \( S \) of \(( -\infty, 0) \) or \(( 0, \infty) \) there exists constants \( K \) and \( \epsilon_0 \) (depending at most on \( u_-, u_+, v_-, v_+, f, g, S \)) such that

\[
\sup_{\xi \in S} (|u_\epsilon(\xi)| + |v_\epsilon(\xi)|) \leq K \quad \text{for} \quad 0 < \epsilon < \epsilon_0.
\]

Proof. Let \( S_+ \subset [a, b], S_- \subset [-b, -a], 0 < a < b < \infty \). Then for \( \epsilon \) sufficiently small \( |\eta^\epsilon| \leq \frac{a}{2} \) and (4.2) yield \( \sup_{\xi \in S_\pm} |v_\epsilon(\xi)| \leq K \). We now need to get a similar estimate on \( u_\epsilon(\xi) \). In case i(a), i(b) of Lemma 2.4, the proof of Lemma 3.3, 3.4, 3.5 yields a uniform in \( \epsilon \) and \( \xi \), \(( -\infty < \xi < \infty) \), bound on \( u_\epsilon(\xi) \) where as in case 0, i(c), ii(a), iii(a), \( u_\epsilon(\xi) \) is monotone so that trivially \( u_- \leq u_\epsilon(\xi) \leq u_+ \) for \( \xi \in (-\infty, \infty) \). Hence the only cases left to search are ii(b),(c), iii(b),(c),(d).

Case ii(b). On \( S_+ \), \( u_\epsilon(\xi) \) is uniformly bounded in \( \epsilon \) and so we need only verify \( S_- \). Let \( \eta \in S_-, \xi \in S_+ \). For \( \epsilon \) sufficiently small \( \eta < \eta^\epsilon < \xi \). Integrate \((P_\epsilon)\) from \( \eta \) to \( \xi \) to obtain

\[
(4.9) \quad \epsilon v_\epsilon'(\xi) - \epsilon v_\epsilon'(\eta) + \int_\eta^\xi \xi v_\epsilon'(\xi) d\xi = g(u_\epsilon(\eta)) - g(u_\epsilon(\xi)).
\]

Since \( v_\epsilon'(\xi) > 0 \) and \( v_\epsilon'(\eta) < 0 \), (4.9) implies

\[
\int_\eta^\xi \xi v_\epsilon'(\xi) d\xi \leq g(u_\epsilon(\eta)) - g(u_\epsilon(\xi)).
\]

and integration by parts yields

\[
(4.10) \quad \xi v_\epsilon(\xi) - \eta v_\epsilon(\eta) - \int_\eta^\xi \xi v_\epsilon'(\xi) d\xi \leq g(u_\epsilon(\eta)) - g(u_\epsilon(\xi)).
\]

Now use (4.1), (4.2) to bound the right-hand side of (4.10) from below

\[
\xi \bar{u} + \frac{\xi(g(\beta) - g(\alpha))}{|\xi - \eta^\epsilon|} - \eta N_0 - N_0(\xi - \eta) \leq g(u_\epsilon(\eta)) - g(u_\epsilon(\xi)).
\]

Since \( \alpha \leq u_\epsilon(\xi) \leq u_+ \), we see \( g(u_\epsilon(\xi)) \leq g(\beta) \). Hence this fact combined with \( |\xi - \eta^\epsilon| \geq \frac{a}{2} \) yields

\[
(4.11) \quad -b|\bar{u}| + \frac{2b(g(\beta) - g(\alpha))}{a} - b N_0 + g(\beta) \leq g(u_\epsilon(\eta)).
\]
Since \( u_\epsilon(\eta) \leq \beta \), (4.11) and the fact that \( g(u) \to -\infty \) as \( u \to -\infty \) show \( u_\epsilon(\eta) \) uniformly bounded in \( \epsilon, \eta \) for \( \epsilon \) sufficiently small, \( \eta \in S_- \).

Case ii(c), iii(b). Proceed as for Case ii(b).

Case iii(c). From the mean value theorem there is \( \zeta \in [1, 2] \) such that
\[
v'\epsilon(\zeta) = v_\epsilon(2) - v_\epsilon(1) \quad \text{and so by (4.2) } \epsilon v'\epsilon(\zeta) \text{ is uniformly bounded. Thus for this } \zeta \text{ and arbitrary } \eta \in S_- \text{ we again derive (4.9) and since } v'\epsilon(\eta) < 0 \text{ we find that}
\]
\[
\epsilon v'\epsilon(\zeta) - \int_\eta^\zeta v_\epsilon(\xi) \, d\xi \leq g(u_\epsilon(\eta)) - g(\alpha).
\]

The same argument as given above for case iii(b) shows \( u_\epsilon(\eta) \) is uniformly bounded in \( \epsilon, \eta \) for \( \epsilon \) sufficiently small, \( \eta \in S_- \).

Case iii(d). Proceed analogously as in Case iii(c).

**Lemma 4.3.** Let \( \{(u_\epsilon(\xi), v_\epsilon(\xi))|0 < \epsilon < 1\} \) be a solution of (\( P_\epsilon \)) as given by Corollary 2.7 when \( u_- < \alpha, u_+ > \beta \). Let \( \xi_-^\epsilon, \xi_+^\epsilon \) denote the points of local minima for \( v_\epsilon(\xi) \) (when they exist). Define \( \bar{u} = \min(u_-, u_+) \),
\[
B^-_\epsilon = u_- - \left( \frac{f'(v_-)}{g'(u_-)} \right)^{1/2} v_- + \left( \frac{f'(v_-)}{g'(u_-)} \right)^{1/2} \left( \bar{v} + \frac{g(\beta) - g(\alpha)}{|\xi_-^\epsilon - \eta^\epsilon|} \right)
\]
\[
B^+_\epsilon = u_+ - \left( \frac{f'(v_+)}{g'(u_+)} \right)^{1/2} v_+ - \left( \frac{f'(v_+)}{g'(u_+)} \right)^{1/2} \left( \bar{v} + \frac{g(\beta) - g(\alpha)}{|\xi_+^\epsilon - \eta^\epsilon|} \right).
\]

Then in the case of Lemma 2.4(with \( \mu = 1, L = \infty \)) we have the following estimates:

In cases 0, i(a),(b),(c), (3.1) holds.

In the remaining cases \( v_\epsilon(\xi) \) satisfies (4.2) and \( u_\epsilon(\xi) \) satisfies
\[
u_- \leq u_\epsilon(\xi) \leq u_+ \text{ in case i(d), ii(a), iii(a).}
\]
\[
B^-_\epsilon \leq u_\epsilon(\xi) \leq B^+_\epsilon \text{ in case ii(b), iii(c).}
\]
\[
u_- \leq u_\epsilon(\xi) \leq B^+_\epsilon \text{ in case ii(c), iii(d).}
\]
\[
B^-_\epsilon \leq u_\epsilon(\xi) \leq B^+_\epsilon \text{ in case iii(b).}
\]

**Lemma 4.4.** Let \( \{(u_\epsilon(\xi), v_\epsilon(\xi))|0 < \epsilon < 1\} \) be a solution of (\( P_\epsilon \)) as given by Corollary 2.7 when \( u_- < \alpha, u_+ > \beta \). Then on any semi-infinite interval
for $0 < \epsilon < \epsilon_0$.

**Lemma 4.5.** Let $\{(u_\epsilon(\xi), v_\epsilon(\xi))|0 < \epsilon < 1\}$ be a solution of $(P_\epsilon)$ as given by Corollary 2.7 when $u_- < \alpha$, $u_+ > \beta$. Then the sequence $(u_\epsilon(\xi), v_\epsilon(\xi))$ possesses a subsequence which converges almost everywhere on $(-\infty, \infty)$ to functions $(u(\xi), v(\xi))$. On compact subsets of $(-\infty, 0) \cup (0, \infty)$ the convergent subsequence is bounded uniformly in $\epsilon$ with uniformly bounded total variation. The limit functions have bounded variation on compact subsets of $(-\infty, 0) \cup (0, \infty)$.

**Lemma 4.6.** The functions $u(\xi), v(\xi)$ defined by Lemma 4.5 satisfy the boundary conditions

$$u(\pm \infty) = u_\pm, v(\pm \infty) = v_\pm.$$
and using Grownwall’s inequality we have

\[
\left| \exp \left( \frac{\xi^2}{2\epsilon} \right) Y'_\epsilon(\xi) \right| \leq \left| \exp \left( \frac{1}{2\epsilon} \right) Y'_\epsilon(1) \right| \exp \left( \frac{R}{\epsilon} \right) (\xi - 1)
\]

and hence

(4.13) \[ |Y'_\epsilon(\xi)| \leq |Y'_\epsilon(1)| \exp \left( \frac{2R\xi - 2R + 1 - \xi^2}{2\epsilon} \right). \]

Note that

\[
\exp \left( \frac{\xi^2}{2\epsilon} \right) Y'_\epsilon(\xi)
= z_1 + \frac{1}{\epsilon} \int_1^\xi F(Y'_\epsilon(\zeta)) \exp \left( \frac{\zeta^2}{2\epsilon} \right) d\zeta
= z_2 + \frac{1}{\epsilon} F(Y'_\epsilon(\xi)) \exp \left( \frac{\xi^2}{2\epsilon} \right) + \frac{1}{\epsilon^2} \int_1^\xi \zeta F(Y'_\epsilon(\zeta)) \exp \left( \frac{\zeta^2}{2\epsilon} \right) d\zeta
\]

and hence

(4.14) \[ Y'_\epsilon(\xi) = z_2 \exp \left( -\frac{\xi^2}{2\epsilon} \right) + \frac{1}{\epsilon} F(Y'_\epsilon(\xi)) \exp \left( \frac{\xi^2}{2\epsilon} \right) \]

Here

(4.15) \[ z_2 \int_1^2 \exp \left( -\frac{\xi^2}{2\epsilon} \right) d\xi
= Y_\epsilon(2) - Y_\epsilon(1) - \frac{1}{\epsilon} \int_1^2 F(Y'_\epsilon(\xi)) d\xi + \frac{1}{\epsilon^2} \int_1^2 \zeta F(Y'_\epsilon(\zeta)) \exp \left( \frac{\zeta^2}{2\epsilon} \right) d\zeta.
\]

Thus from (4.14) we have

(4.16) \[ |Y'_\epsilon(1)| \leq |z_2| \exp \left( -\frac{1}{2\epsilon} \right) + \frac{1}{\epsilon} |F(Y'_\epsilon(1))| \]

\[ \leq |z_2| \exp \left( -\frac{1}{2\epsilon} \right) + \frac{\text{const}}{\epsilon}. \]
From (4.15) and the inequality
\[ \int_1^2 \exp \left( -\frac{x^2}{2\epsilon} \right) \, dx \geq \exp \left( -\frac{2}{\epsilon} \right) \]
we see that
\[ |z_2| \leq \left( \text{const} + \frac{\text{const}}{\epsilon} + \frac{\text{const}}{\epsilon^2} \exp \left( \frac{2}{\epsilon} \right) \right) \exp \left( \frac{2}{\epsilon} \right) \]
and hence by (4.16) that
\[ |Y'_\epsilon(1)| \leq \frac{\text{const}}{\epsilon^2} \exp \left( \frac{7}{2\epsilon} \right). \] (4.17)
Now insert (4.17) into (4.13) to find that
\[ |Y'_\epsilon(\xi)| \leq \frac{\text{const}}{\epsilon^2} \left( \frac{2R\xi - 2R + 8 - \xi^2}{2\epsilon} \right). \] (4.18)
Thus for \( \xi > R + (R^2 - 2R + 8)^{1/2} \) (4.18) shows that \( |Y'_\epsilon(\xi)| \to 0 \) as \( \epsilon \to 0^+ \). Recalling that \((u_\epsilon(\xi), v_\epsilon(\xi))\) converges pointwise to \((u(\xi), v(\xi))\), we see \((u(\xi), v(\xi))\) must be constants for \( \xi > R + (R^2 - 2R + 8)^{1/2} \). Since for any \( \epsilon > 0 \) \( \lim_{\xi \to \infty} u_\epsilon(\xi) = u_+ \), \( \lim_{\xi \to \infty} v_\epsilon(\xi) = v_+ \), these constants must be \( u_+ \) and \( v_+ \). A similar argument works for \( \xi = -\infty \).

**Corollary 4.7.** The functions \( u(\xi), v(\xi) \) defined by Lemma 4.5 satisfy the conditions
\[ (u(\xi), v(\xi)) = \begin{cases} (u_-, v_-), & \xi < -M, \\ (u_+, v_+), & \xi > M \end{cases} \]
for some positive constant \( M \).

**Lemma 4.8.** The functions \((u(\xi), v(\xi))\) defined by Lemma 4.5 satisfy
\[ -\xi u' - f(v)' = 0, \]
\[ -\xi v' - g(u)' = 0 \] (4.19)
in the sense of distributions at any \( \xi \neq 0 \).
At any point \( \xi_0 \neq 0 \) of discontinuity of \((u(\xi), v(\xi))\) the Rankine-Hugoniot jump conditions are satisfied:
\[ -\xi_0 (u(\xi_0^+) - u(\xi_0^-)) - (f(v(\xi_0^+)) - f(v(\xi_0^-))) = 0, \]
\[ -\xi_0 (v(\xi_0^+) - v(\xi_0^-)) - (g(u(\xi_0^+)) - g(u(\xi_0^-))) = 0. \] (4.20)
Proof. By Lemma 4.5 there exists a sequence of solutions of $(P_\epsilon)$ which converges bounded almost everywhere on any compact subset of $(0, \infty) \cup (-\infty, 0)$. Hence if we multiply $(P_\epsilon)$ by $C^\infty$ test functions with compact support excluding $\xi = 0$, integrate by parts, pass to the limits as the relevant sequence of $\epsilon$’s goes to zero, and use the Lebesgue dominated convergence theorem, we obtain (4.19). Equation (4.19) follows from (4.18) in the standard manner.

**Definition 4.9.** $u, v$ is a distributional solution of (4.19) at $\xi = 0$ if

\[
\begin{align*}
\lim_{\xi \to 0^{-}} f(v(\xi)) &= \lim_{\xi \to 0^{+}} f(v(\xi)), \\
\lim_{\xi \to 0^{-}} g(u(\xi)) &= \lim_{\xi \to 0^{+}} g(u(\xi)).
\end{align*}
\]  

(4.21)

**Lemma 4.10.** Assume that

\[
\frac{1}{|u|} \left| \int_{\beta}^{u} g(\xi) d\xi \right| \to \infty \text{ as } |u| \to \infty.
\]

Then $\{u_\epsilon(\xi)\}$ has absolutely equicontinuous integrals and the functions $u(\xi), v(\xi)$ defined by Lemma 4.5 are locally integrable in $(-\infty, \infty)$.

Proof. From (4.1), $|v_\epsilon(\xi)|$ is locally integrable. Since a subsequence of $v_\epsilon(\xi)$ converges to $v(\xi)$, Fatou’s theorem implies $v(\xi)$ is locally integrable. To show locally integrability of $u(\xi)$, we will show at first $\{u_\epsilon(\xi)\}$ have absolutely equicontinuous integral. In case i(d), ii(a), iii(a) of Lemma 2.4 there is nothing to prove since $u_\epsilon(\xi)$ is monotone and hence uniformly bounded in $\xi, \epsilon$. Theorem 3.8 implies that Case 0, i(a, b, c) were covered. We need only prove Case ii(b, c), iii(b, c, d). Consider ii(c). Given any interval $(l_1, l_2)$ we either

(I) $(l_1, l_2) = (l_1, t_\epsilon) \cup [t_\epsilon, l_2)$ where $(l_1, t_\epsilon)$ if $u_- \leq u_\epsilon(\xi) \leq \beta$ and $[t_\epsilon, l_2)$ if $\beta \leq u_\epsilon, u_\epsilon(t_\epsilon) = \beta$,

(II) $u_\epsilon \geq \beta$ on $(l_1, l_2)$, or

(III) $u_\epsilon(\xi) \leq \beta$ on $(l_1, l_2)$.

First we consider (I). Multiply $(P_\epsilon)_1$ by $g(u)$ and $(P_\epsilon)_2$ by $f(v)$ and add. If we define

$\eta(u, v) = F(v) + \int_{\beta}^{u} g(\xi) d\xi$, $F'(v) = f(v)$ and $\eta_\epsilon(\xi) = \eta(u_\epsilon(\xi), v_\epsilon(\xi))$ we see that

\[
(4.23) \quad \epsilon \eta''_\epsilon(\xi) + \xi \eta'_\epsilon(\xi) + (f(v)g(u))' - \epsilon(u')^2 g'(u) - \epsilon f'(v)(v')^2 = 0.
\]
Let $\bar{\eta} = \max\{\eta(u_-, v_-), \eta(u_+, v_+)\}$. On any subinterval $(s_1, s_2) \subset [t_\epsilon, l_2)$ set

$$\xi_\epsilon = \begin{cases} 
\sup\{\xi \in [t_\epsilon, s_1] | \eta_\epsilon(\xi) \leq \bar{\eta}\} & \text{if } \eta_\epsilon(s_1) > \bar{\eta}, \\
\inf\{\xi \in (s_1, s_2) | \eta_\epsilon(\xi) \geq \bar{\eta}\} & \text{if } \eta_\epsilon(s_1) \leq \bar{\eta}
\end{cases}$$

and

$$\theta_\epsilon = \begin{cases} 
\inf\{\xi \in (s_2, l_2) | \eta_\epsilon(\xi) \leq \bar{\eta}\} & \text{if } \eta_\epsilon(s_2) > \bar{\eta}, \\
\sup\{\xi \in (s_1, s_2) | \eta_\epsilon(\xi) \geq \bar{\eta}\} & \text{if } \eta_\epsilon(s_2) \leq \bar{\eta}.
\end{cases}$$

Observe that $\eta_\epsilon' (\xi_\epsilon) \geq 0, \eta_\epsilon' (\theta_\epsilon) \leq 0$ and

$$(4.24) \quad \int_{s_1}^{s_2} (\eta_\epsilon(\xi) - \bar{\eta}) \, d\xi \leq \int_{\xi_\epsilon}^{\theta_\epsilon} (\eta_\epsilon(\xi) - \bar{\eta}) \, d\xi = -\int_{\xi_\epsilon}^{\theta_\epsilon} \xi_\epsilon' \eta_\epsilon' (\xi) \, d\xi.$$ 

Thus if we integrate (4.23) over $(\xi_\epsilon, \theta_\epsilon)$ and use (4.24) we see that

$$(4.25) \quad \int_{s_1}^{s_2} (\eta_\epsilon(\xi) - \bar{\eta}) \, d\xi + \epsilon \int_{\xi_\epsilon}^{\theta_\epsilon} ((u_\epsilon')^2 g'(u_\epsilon) + f'(v_\epsilon)(v_\epsilon')^2) \, d\xi$$

$$\leq f(v_\epsilon(\theta_\epsilon)) - f(v_\epsilon(\xi_\epsilon)) g(u_\epsilon(\xi_\epsilon)).$$

By the definitions of $\theta_\epsilon, \xi_\epsilon, \eta(u_\epsilon(\theta_\epsilon), v_\epsilon(\theta_\epsilon))$ and $\eta(u_\epsilon(\xi_\epsilon), v_\epsilon(\xi_\epsilon))$ are uniformly bounded from above and since $u_\epsilon(\theta_\epsilon) \geq \beta, \eta$ is convex at these values. This implies $u_\epsilon(\theta_\epsilon), v_\epsilon(\theta_\epsilon), u_\epsilon(\xi_\epsilon), v_\epsilon(\xi_\epsilon)$ are uniformly bounded in $\epsilon$. Hence the right-hand side of (4.25) is bounded by a constant $K = K(f, g, u_\epsilon, v_\epsilon)$ independent of $\epsilon$. Now since $\frac{1}{u} \int_B g(s) \, ds \to \infty$ as $u \to \infty$, for any $\delta > 0$ there is $u_0 \geq \beta$ such that

$$\frac{u}{\eta(u, v)} < \frac{\delta}{2K} \text{ for all } u \geq u_0.$$ 

Set $l(\delta) = \frac{\delta}{(|u_-|+\beta+u_0+\frac{\delta}{2K})}$. Fix $s_1, s_2, 0 < s_2 - s_1 < l(\delta)$. Note that for any $s_1, s_2, s_1 \in (l_1, t_\epsilon), s_2 \in (t_\epsilon, l_2),$

$$\int_{s_1}^{s_2} u_\epsilon(\xi) \, d\xi = \int_{s_1}^{t_\epsilon} u_\epsilon(\xi) \, d\xi + \int_{t_\epsilon}^{s_2} u_\epsilon(\xi) \, d\xi$$

$$\leq \beta(t_\epsilon - s_1) + \int_{t_\epsilon}^{s_2} (u_0 + \frac{\delta}{2K} \eta(u_\epsilon(\xi), v_\epsilon(\xi))) \, d\xi$$

$$\leq \beta(t_\epsilon - s_1) + (s_2 - t_\epsilon)u_0 + \frac{\delta}{2K} \int_{t_\epsilon}^{s_2} \eta(u_\epsilon(\xi), v_\epsilon(\xi)) \, d\xi.$$
Using (4.24) with \( s_2 = s_2, s_1 = t_e \),

\[
\int_{s_1}^{s_2} u_e(\xi) \, d\xi \leq \beta(t_e - s_1) + (s_2 - t_e)u_0 + \frac{\delta}{2K}(K + \bar{\eta}(s_2 - s_1))
\]

\[
\leq (s_2 - s_1)(\beta + u_0 + \frac{\bar{\eta}\delta}{2K}) + \frac{\delta}{2}
\]

\[
\leq \delta.
\]

If \( s_1, s_2 \geq t_e \),

\[
\int_{s_1}^{s_2} u_e(\xi) \, d\xi \leq \int_{s_1}^{s_2} (u_0 + \frac{\delta}{2K}\eta(u_e(\xi), v_e(\xi))) \, d\xi \leq \delta
\]

and if \( s_1, s_2 \leq t_e \)

\[
\int_{s_1}^{s_2} u_e(\xi) \, d\xi \leq \beta(s_2 - s_1) \leq \delta.
\]

Also since \( u_e(\xi) \geq u_- \) we have

\[
\int_{s_1}^{s_2} u_e(\xi) \, d\xi \geq u_-(s_2 - s_1) \geq -|u_-|(s_2 - s_1) \geq -\delta.
\]

Thus we proved that

\[
\left| \int_{s_1}^{s_2} u_e(\xi) \, d\xi \right| \leq \delta \text{ if } 0 < s_2 - s_1 < l(\delta).
\]

Now using Vitali’s theorem, \( u \) is locally integrable.

**Lemma 4.11.** The four limits which appear in (4.21) always exist and (4.21) is always satisfied. Equation (4.21) is satisfied if the sequence \( \{\int_{0}^{\xi} v_e(\xi) \, d\xi\} \) is absolutely equicontinuous. Furthermore in general

\[
g(\beta) - g(\alpha) \leq \lim_{\theta \to 0^+} g(u(\theta)) - \lim_{\xi \to 0^-} g(u(\xi)) \leq 0.
\]
Proof. Let \(\{(u_\epsilon(\xi), v_\epsilon(\xi))\}\) denote the convergent subsequence of Lemma 4.5. Note that since \(u_\epsilon(\xi), v_\epsilon(\xi)\) are piecewise monotone in \((-\infty, \infty)\), the limit functions \(u(\xi), v(\xi)\) are also monotone and hence the set of points of continuity of \(u, v\) is dense in any finite \(\xi\)-interval. Let \(\zeta\) and \(\theta\) be points of continuity of \(u(\xi), v(\xi)\), \(\zeta < 0 < \theta\). From the mean value theorem for every small \(\epsilon > 0\) we can find \(\zeta_\epsilon \in [\zeta - \epsilon^{1/2}, \zeta], \theta_\epsilon \in [\theta, \theta + \epsilon^{1/2}]\) such that

\[
\epsilon^{1/2} v_\epsilon'(\zeta_\epsilon) = v_\epsilon(\zeta) - v_\epsilon(\zeta - \epsilon^{1/2}), \quad \epsilon^{1/2} u_\epsilon'(\zeta_\epsilon) = u_\epsilon(\zeta) - u_\epsilon(\zeta - \epsilon^{1/2}),
\]

\[
\epsilon^{1/2} v_\epsilon'(\theta_\epsilon) = v_\epsilon(\theta) - v_\epsilon(\theta - \epsilon^{1/2}), \quad \epsilon^{1/2} u_\epsilon'(\theta_\epsilon) = u_\epsilon(\theta) - u_\epsilon(\theta - \epsilon^{1/2}).
\]

By Lemma 2.4 there are constants \(K_\theta, K_\zeta\) such that

\[
|\epsilon^{1/2} v_\epsilon'(\zeta_\epsilon)| \leq K_\zeta, \quad |\epsilon^{1/2} u_\epsilon'(\zeta_\epsilon)| \leq K_\zeta,
\]

\[
|\epsilon^{1/2} v_\epsilon'(\theta_\epsilon)| \leq K_\theta, \quad |\epsilon^{1/2} u_\epsilon'(\theta_\epsilon)| \leq K_\theta.
\]

for \(\epsilon\) sufficiently small. Now we integrate \((P_\epsilon)\) on \((\zeta_\epsilon, \theta_\epsilon)\) obtaining

\[
\epsilon u_\epsilon'(\theta_\epsilon) - \epsilon u_\epsilon'(\zeta_\epsilon) + \theta_\epsilon u_\epsilon(\theta_\epsilon) - \zeta_\epsilon u_\epsilon(\zeta_\epsilon) - \int_{\zeta_\epsilon}^{\theta_\epsilon} u_\epsilon(\xi) \, d\xi = f(v(\zeta_\epsilon)) - f(v(\theta_\epsilon)),
\]

\[
\epsilon v_\epsilon'(\theta_\epsilon) - \epsilon v_\epsilon'(\zeta_\epsilon) + \theta v_\epsilon(\theta_\epsilon) - \zeta v_\epsilon(\zeta_\epsilon) - \int_{\zeta_\epsilon}^{\theta_\epsilon} v_\epsilon(\xi) \, d\xi = g(u(\zeta_\epsilon)) - g(u(\theta_\epsilon)).
\]

Now let \(\epsilon \to 0^+\) in (4.27). Since \(\theta, \zeta\) are points of continuity of \(u, v\) we find by virtue of (4.26) and the Vitali’s theorem that

\[
\theta u(\theta) - \zeta u(\zeta) + f(v(\theta)) - f(v(\zeta)) = \lim_{\epsilon \to 0^+} \int_{\zeta_\epsilon}^{\theta_\epsilon} u_\epsilon(\xi) \, d\xi
\]

\[
\theta v(\theta) - \zeta v(\zeta) + g(u(\theta)) - g(u(\zeta)) = \lim_{\epsilon \to 0^+} \int_{\zeta_\epsilon}^{\theta_\epsilon} v_\epsilon(\xi) \, d\xi
\]

Since the limits on the left hand side of (4.27) exists, we have from (4.1)

\[
\lim_{\epsilon \to 0^+} \int_{\zeta_\epsilon}^{\theta_\epsilon} v_\epsilon(\xi) \, d\xi := S(\zeta, \theta)
\]
satisfies

\[
\tilde{v}(\xi - \theta) + (g(\beta) - g(\alpha)) \leq S(\xi, \theta) \leq N_0(\xi - \theta).
\]

By Lemma 4.4 for fixed \( \xi < 0 \), \( S(\xi, \theta) \) is continuous in \( \theta, \theta > 0, |\theta| \) small\ and for fixed \( \theta > 0 \), \( S(\xi, \theta) \) is continuous in \( \xi, \xi < 0, |\xi| \) small. Now since \( |u(\xi)| \) may be infinite only at \( \xi = 0 \) pointwise limits of ii(b, c), iii(b, c, d) of Lemma 2.4 shows that if \( |u(0)| = \infty \), \( u \) must one of these shape shown in figure.

In all these cases (I), (II), (III) we see that

\[
|\xi u(\xi)| \leq \int_{\xi}^{\theta} |u(\xi)| \, d\xi,
\]
\[
|\theta u(\theta)| \leq \int_{\xi}^{\theta} |u(\xi)| \, d\xi
\]

But since \( u(\xi) \) is locally integrable,

\[
\lim_{\xi \to 0^-} \xi u(\xi) = \lim_{\theta \to 0^+} \theta u(\theta) = \lim_{\xi \to 0^-} \int_{\xi}^{\theta} u(\xi) \, d\xi = 0
\]

Since \( v(\xi) \) has the shape of (I) near \( \xi = 0 \) and \( v \) is locally integrable

\[
\lim_{\xi \to 0^+} \xi v(\xi) = \lim_{\theta \to 0^+} \theta v(\theta) = 0
\]

Now let \( \theta \to 0^+, \xi \to 0^- \) along a sequence of points of continuity of \( u, v \) and possibly extract a further subsequence such that \( S(\xi, \theta) \) converges we find that

\[
\lim_{\theta \to 0^+} f(v(\theta)) - \lim_{\xi \to 0^-} f(v(\xi)) = 0,
\]
\[
\lim_{\theta \to 0^+} g(u(\theta)) - \lim_{\xi \to 0^-} g(u(\xi)) = \lim_{\theta \to 0^+} S(\xi, \theta).
\]

Moreover if \( \int_{0}^{\xi} v(\xi) \, d\xi \) is absolutely equicontinuous, the Vitali’s theorem implies

\[
\lim_{\xi \to 0^-} S(\xi, \theta) = 0.
\]

In general, the bounds on \( S(\xi, \theta) \) shows that

\[
g(\beta) - g(\alpha) \leq \lim_{\theta \to 0^+} g(u(\theta)) - \lim_{\xi \to 0^-} g(u(\xi)) \leq 0.
\]
THEOREM 4.12. The functions $u(\xi)$, $v(\xi)$ defined by Lemma 4.5 is a solution of the Riemann problem provided

$$\lim_{\xi \to 0^-} g(u(\xi)) = \lim_{\xi \to 0^+} g(u(\xi)).$$

Proof. Use Lemma 4.11.

References

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