EXOTIC SYMPLECTIC STRUCTURES ON $S^3 \times \mathbb{R}$

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ABSTRACT. We construct exotic symplectic structures on $S^3 \times \mathbb{R}$ which is obtained by the symplectic sum of two smooth symplectic four-manifolds with exotic symplectic structures, each of which is diffeomorphic to $\mathbb{R}^4$.

1. Introduction

Let $\omega_0$ be the standard symplectic structure on $\mathbb{R}^{2n}$ and $L \subset \mathbb{R}^{2n}$ be a closed Lagrangian submanifold. In [3], Gromov have shown the following theorem:

**Theorem (Gromov).** As a cohomology class $[\omega_0]$ is non-zero in $H^2(\mathbb{R}^{2n}, L; \mathbb{R})$. The form $\omega_0$ has a potential $\psi$ on $\mathbb{R}^{2n}$, i.e., $\omega_0 = d\psi$. Furthermore, $[\psi|_L] \neq 0$ in $H^1(L; \mathbb{R})$.

The Lagrangian submanifold $L$ in a $2n$-dimensional symplectic manifold $M$ is called exact(non-exact) if the restriction to the Lagrangian $L$ of the potential is exact(non-exact). Thus, in the above Theorem, $L$ is a non-exact Lagrangian in $\mathbb{R}^{2n}$.

Gromov have also proved that there are no exact Lagrangian sub-varieties in $\mathbb{R}^{2n}$, for the standard symplectic structure. Recently, Bates and Peschke [1] have explicitly endowed a manifold $M$ diffeomorphic to $\mathbb{R}^4$ with a symplectic form $\omega$ admitting a Lagrangian torus $T$ such that $[\omega] = 0$ in $H^2(M, T; \mathbb{R})$. Hence $T$ is an exact Lagrangian. By Gromov’s theorem, $(M, \omega)$ does not symplectically embed in $(\mathbb{R}^4, \omega_0)$, such a structure $\omega$ is called an exotic symplectic structure on $M$.

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Let $M_i$ ($i = 1, 2$) be smooth symplectic four-manifolds diffeomorphic to $\mathbb{R}^4$ with symplectic forms admitting Lagrangian tori $(T'_i)$ ($i = 1, 2$).

In section 2, we introduce the symplectic sum of these two manifolds and construct symplectic forms $\omega_M$ on the sum $M = M_1 \# \psi M_2$ from symplectic forms on the $M_i$ ($i = 1, 2$). We first show that

**Lemma 2.3.** $M = M_1 \# \psi M_2 \cong (M_1 - S_1 - K) \cup \varphi ((M_2 - S_2 - j_2(D^2))) \cong S^3 \times \mathbb{R}$, where $S_i$ are the interior surfaces of $S_i$ on $(T_1)$ with the boundaries $S_i^1 = j_i(\partial D^2)$ ($i = 1, 2$). Hence $H^1(T'_2; \mathbb{R}) \cong H^2(M, T'_2; \mathbb{R})$ is an isomorphism, where $T'_2$ is a Lagrangian surface of genus 2 in $M$.

In section 3, we show the process of constructing symplectic forms $\omega'_M$ on $M = M_1 \# \psi M_2 \cong S^3 \times \mathbb{R}$ from exotic symplectic forms on two smooth symplectic four-manifolds $M_i$ ($i = 1, 2$) diffeomorphic to $\mathbb{R}^4$.

In section 4, we get the following two Lemmas 4.1 and 4.2 from each case of manifolds $(M, \omega_M)$ and $(M, \omega'_M)$:

**Lemma 4.1.** The symplectic forms $\omega_M$ admit a non-exact Lagrangian surface $T'_2$ of genus 2 in $M$ and hence $[\omega_M] \neq 0$ in $H^2(M, T'_2; \mathbb{R})$.

**Lemma 4.2.** The symplectic forms $\omega'_M$ admit an exact Lagrangian surface $T_2$ of genus 2 in $M$ and hence $[\omega'_M] = 0$ in $H^2(M, T_2; \mathbb{R})$.

By the Lemmas 4.1 and 4.2, we can get the following Theorem 4.3.

**Theorem 4.3.** The symplectic forms $\omega_M$ on the symplectic sum $M$ of two smooth symplectic four-manifolds $M_i$ ($i = 1, 2$) diffeomorphic to $\mathbb{R}^4$ with symplectic forms admitting non-exact Lagrangian tori $(T'_i)$ ($i = 1, 2$) admit a non-exact Lagrangian surface $T'_2$ of genus 2 and $[\omega_M] \neq 0$ in $H^2(M, T'_2; \mathbb{R})$.

On the other hand, the symplectic forms $\omega'_M$ on the symplectic sum $M$ of two smooth symplectic four-manifolds $M_i$ ($i = 1, 2$) diffeomorphic to $\mathbb{R}^4$ with symplectic forms admitting exact Lagrangian tori $T'_i$ ($i = 1, 2$) admit an exact Lagrangian surface $T_2$ of genus 2 and $[\omega'_M] = 0$ in $H^2(M, T_2; \mathbb{R})$. Therefore, $(M, \omega'_M)$ does not symplectically diffeomorphic to $(M, \omega_M)$.

2. Symplectic sums

Let $M_i$ ($i = 1, 2$) be smooth symplectic four-manifolds which are diffeomorphic to $\mathbb{R}^4$. Let $\mathbb{R}^4$ be thought of as $\mathbb{R}^2 \times \mathbb{R}^2$ and let $(r, \theta), (s, \phi)$ be...
polar coordinates on each factor. That is, if \((x_1, x_2)\) and \((y_1, y_2)\) are rectangular coordinates on each factor of \(\mathbb{R}^2 \times \mathbb{R}^2\), then \(x_1 = r \cos \theta, x_2 = r \sin \theta, y_1 = s \cos \phi, y_2 = s \sin \phi\). Suppose that \(\mathbb{R}^4\) has a standard symplectic structure \(\omega_{\mathbb{R}^4} = \sum_{i=1}^2 dx_i \wedge dy_i\).

Let \(T_1 = \{(x_1, x_2, y_1, y_2) \in \mathbb{R}^4 | x_1^2 + x_2^2 = \frac{r^2}{2}, y_1^2 + y_2^2 = \frac{s^2}{2} \} = \{(\sqrt{\frac{2}{r}} \cos \theta, \sqrt{\frac{2}{s}} \cos \phi, \sqrt{\frac{2}{r}} \sin \theta, \sqrt{\frac{2}{s}} \sin \phi) \in \mathbb{R}^4 | 0 \leq \theta < 2\pi, 0 \leq \phi < 2\pi \}. \) Let \(j : T_1 \to \mathbb{R}^4\) be an embedding defined by \(j(r \cos \theta, r \sin \theta, s \cos \phi, s \sin \phi) = (r \cos \theta, s \cos \phi, r \sin \theta, s \sin \phi)\). Then \(T'_1 = j(T_1)\) is a torus defined by \(x_1^2 + y_1^2 = \frac{r^2}{2}\) and \(x_2^2 + y_2^2 = \frac{s^2}{2}\), and a closed Lagrangian in \(\mathbb{R}^4\) with respect to \(\omega_{\mathbb{R}^4}\) since \(j^* \omega_{\mathbb{R}^4} |_{T'_1} = j^* \omega_{\mathbb{R}^4}\) and

\[
\begin{align*}
    j^* \omega_{\mathbb{R}^4}(m) &\left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}\right) \\
    &= \omega_{\mathbb{R}^4}(j(m))(dj\left(\frac{\partial}{\partial \theta}|_m\right), dj\left(\frac{\partial}{\partial \phi}|_m\right)) \\
    &= (dx_1 \wedge dy_1 + dx_2 \wedge dy_2)(j(m)) \\
    &= \left(-r \sin \alpha \left|_{j(m)}\right. \frac{\partial}{\partial x_1} - r \cos \alpha \left|_{j(m)}\right. \frac{\partial}{\partial y_1}, \right. \\
    &\left.- s \sin \beta \left|_{j(m)}\right. \frac{\partial}{\partial x_2} + s \cos \beta \left|_{j(m)}\right. \frac{\partial}{\partial y_2}\right) \\
    &= -r \sin \alpha \cdot 0 - 0 \cdot r \cos \alpha + 0 \cdot s \cos \beta + s \sin \beta \cdot 0 \\
    &= 0
\end{align*}
\]

for all \(m = (r \cos \theta, r \sin \theta, s \cos \phi, s \sin \phi) \in T_1\).

By Gromov’s theorem in section 1, \([\omega_{\mathbb{R}^4}] \neq 0\) in \(H^2(\mathbb{R}^4, T'_1; \mathbb{R})\) and \([\sum_{i=1}^2 x_i dy_i|_{T'_1}] \neq 0\) in \(H^1(T'_1; \mathbb{R})\). If we take \(\phi_i\) as diffeomorphism from \(M_i\) to \(\mathbb{R}^4\) such that \(\phi_i^{-1}(T'_1) = (T'_1)^i\) and if we set \(\omega_{M_i} = \phi_i^* \omega_{\mathbb{R}^4}\) as symplectic structures on \(M_i\) \((i = 1, 2)\), then \((T'_1)^i\) are closed Lagrangian tori in \(M_i\), since \(\omega_{M_i}|_{(T'_1)^i} = \phi_i^* \omega_{\mathbb{R}^4}|_{(T'_1)^i} = \omega_{\mathbb{R}^4}|_{T'_1} = 0\). Moreover, \((T'_1)^i\) are non-exact Lagrangian tori in \(M_i\) since \([\phi_i^{-1}(\sum_{i=1}^2 x_i dy_i)|_{(T'_1)^i}] = [\sum_{i=1}^2 x_i dy_i|_{T'_1}] \neq 0\) in \(H^1((T'_1)^i; \mathbb{R})\). By isomorphisms \(H^1((T'_1)^i; \mathbb{R}) \cong H^2(M_i, (T'_1)^i; \mathbb{R})\), \([\omega_{M_i}] \neq 0\) in \(H^2(M_i, (T'_1)^i; \mathbb{R})\).

Let \(D^2\) be the standard closed 2-dimensional disk of radius \(\sqrt{r}\) with symplectic structure \(\omega_{D^2} = dx_1 \wedge dy_1\). Let \(h : (D^2, \partial D^2) \to (\mathbb{R}^4, T'_1)\) be
defined by \( h(x_1, y_1) = \left( \frac{x_1}{\sqrt{2}}, \frac{y_1}{\sqrt{2}}, \frac{y_2}{\sqrt{2}}, -\frac{x_2}{\sqrt{2}} \right) \), and let \( j_i = \varphi_i^{-1} \circ h : (D^2, \partial D^2) \rightarrow (M_i, (T_i^1))^i \). Then \( j_i \) are symplectic embeddings satisfying \( j_i(\partial D^2) \subset (T_i^1)^i \) and \((j_i(D^2) - j_i(\partial D^2)) \cap (T_i^1)^i = \emptyset \) (\( i = 1, 2 \)) since \( j_i^*\omega_{M_i} = j_i^*\varphi_i^*\omega_{\mathbb{R}^4} = (\varphi_i \circ j_i)^*\omega_{\mathbb{R}^4} = h^*\omega_{\mathbb{R}^4} \) and

\[
\begin{align*}
 h^*\omega_{\mathbb{R}^4} &= h^*(dx_1 \wedge dy_1 + dx_2 \wedge dy_2) \\
 &= \frac{1}{\sqrt{2}} dx_1 \wedge \frac{1}{\sqrt{2}} dy_1 + \frac{1}{\sqrt{2}} dy_1 \wedge (-\frac{1}{\sqrt{2}}) dx_1 \\
 &= \frac{1}{2} dx_1 \wedge dy_1 - \frac{1}{2} dy_1 \wedge dx_1 \\
 &= dx_1 \wedge dy_1 \\
 &= \omega_{D^2}.
\end{align*}
\]

We can choose a fiber-orientation reversing bundle isomorphism \( \psi : v_1 \rightarrow v_2 \). We choose fiber metrics on \( v_i \) such that \( \psi \) is isometric. Let \( v_i^0 \) be disk bundles in \( v_i \) (\( i = 1, 2 \)). Then there is an orientation-preserving diffeomorphism \( \varphi = \iota \circ \psi : v_1^0 - j_1(D^2) \rightarrow v_2^0 - j_2(D^2) \), where the map \( \iota : v_2^0 - \{0 - section\} \rightarrow v_2^0 - \{0 - section\} \) is defined by \( \iota(x) = (\frac{1}{\pi |x|} - 1)^{1/2} x \).

Now we construct suitable models for tubular neighborhoods of the sub-manifolds \( j_i(D^2) \) in \( M_i \) (\( i = 1, 2 \)). Let \( v_i \) denote the \( SO(2) \)-vector bundles over \( D^2 \) and let \( v_i^0 \) denote the sub-disk bundles of radius \( \pi^{-1/2} \) (\( i = 1, 2 \)). Let \( \pi : S \rightarrow D^2 \) be the 2-sphere bundle obtained by gluing together \( v_1^0 \) and \( v_2^0 \) using \( \iota \) defined in the above statement. We may take the sphere bundle \( S \) over \( D^2 \) as \( D^2 \times S^2 \). Let \( i_0, i_{\infty} : D^2 \rightarrow S \) be 0-sections of \( v_0^0 \) and \( v_2^0 \) with images \( D_0 \) and \( D_{\infty} \), respectively. Thus, \( v^0_1 = S - D_{\infty} \).

Considering cylindrical polar coordinates \((\theta, x_3)\) on \( S^2 - \{(0, 0, \pm 1)\} \) where \( 0 \leq \theta < 2\pi \) and \(-1 \leq x_3 \leq 1 \), we can take a symplectic form \( \omega_{S^2} \) on \( S^2 \) as the area form \( \omega_{S^2} = d\theta \wedge dx_3 \) induced by the Euclidean metric. Hence we may choose a closed 2-form \( \eta \) on the sphere bundle \( S \cong D^2 \times S^2 \) over \( D^2 \) as \( \omega_{S^2} \). Then \( \eta \) has the following properties: \( \iota^*_0 \eta = \eta|_{i_0(D^2)} = \eta|_{D_0} = 0 \) and \( \eta|_{S^2} = d\theta \wedge dx_3 \) is the symplectic form. By the method of Thurston[8], we can thus construct the set of symplectic forms on \( S \) as \( \{\omega_t = \pi^*\omega_{D^2} + t \cdot \eta \mid 0 < t \leq t_1 \} \) for some sufficiently small constant \( t_1 > 0 \).

On the other hand, there is a smooth orientation-preserving embedding \( f : v^0_1 \rightarrow M_1 \) (into any preassigned neighborhood of \( j_1(D^2) \)) with \( f \circ i_0 = j_1 \).
And $f|_{D_0} : (D_0, \omega_t) \to (M_1, \omega_{M_1})$ is symplectic, since $i_0^* \omega_t = i_0^* \pi^* \omega_{D^2} + t \cdot i_0^* \eta = (\pi \circ i_0)^* \omega_{D^2} = \omega_{D^2}$, $\tilde{f} \circ i_0 = j_1$ and $j_1$ is symplectic. Thus we get the following Theorem 2.1 which is the same result as Gompf’s.

**Theorem 2.1.** Let $(v_1^0, \omega_t)$, $(M_1, \omega_{M_1})$, $D_0$ and $f : v_1^0 \to M_1$ be the same as above. Then there is a compactly supported isotopy rel $D_0$ from $f$ to an embedding $\tilde{f} : v_1^0 \to M_1$ that is symplectic in a neighborhood of $D_0$.

**Proof.** It can be proved by the same way as the proof of Lemma 2.1 in [2].

Weinstein’s integral operator $I : \Omega^2(v_1^0) \to \Omega^1(v_1^0)$ is defined by $I(\eta) = \int_0^1 \pi_s^*(X_s \cdot \eta) ds$, where $\pi_s : v_1^0 \to v_1^0 (0 \leq s \leq 1)$ is a multiplication by $s$ in this bundle structure, $X_s = \frac{d}{ds} \pi_s$ the corresponding vector field, and $\cdot$ denotes contraction. The key property of $I$ is that if $\eta$ satisfies $d\eta = 0$ and $i_0^* \eta = 0$, then $dI(\eta) = 0$. Set $\phi = I(\eta)$, and define $Y_t$ by $Y_t \cdot \omega_t = -\phi, 0 < t \leq t_1$. Then $Y_t(0 < t \leq t_1)$ is a time-dependent vector field on $v_1^0$ that vanishes on $D_0$ and $SO(2)$-invariant. For any $SO(2)$-invariant compact subset $K \subset v_1^0$ and fixed $t_0 \in (0, t_1]$, $Y_t$ integrates to an $SO(2)$-equivariant flow $F : K \times J \to v_1^0$, where $J$ is some neighborhood of $t_0$ in $(0, t_1]$ and $F_{t_0} = id_K$. Since $\frac{d}{dt}(F_t^* \omega_t) = dF_t^* (Y_t \cdot \omega_t) + F_t^* (\frac{d}{dt} \omega_t) = -F_t^* d\phi + F_t^* \eta = -F_t^* \eta + F_t^* \eta = 0$, $F_t^* \omega_t$ is independent of $t$.

For $x \in v_1^0$, let $D(x)$ be the closed disk in the fiber $\pi^{-1}(\pi(x))$ that is bounded by the $SO(2)$-orbit of $x$. Let $A(x) = \int_{D(x)} \eta$ be the $\eta$-area of $D(x)$. Then $A : v_1^0 \to [0, 1)$ is a smooth, $SO(2)$-invariant, proper surjection that increases radially. The $\omega_t$-area of $D(x)$ is given by $\int_{D(x)} \omega_t = \int_{D(x)} (\pi^* \omega_{D^2} + t \cdot \eta) = t \int_{D(x)} \eta = t \cdot A(x)$. Fix $x \in v_1^0$ and $t_0 \in (0, t_1]$, and integrate $Y_t$ as above to obtain a flow of $D(x)$ with $F_{t_0} = id_{D(x)}$. Let $x(t) = F_t(x)$ be the trajectory of $x$, with $x(t_0) = x$. Since $F$ is $SO(2)$-equivariant, $\partial F_t D(x) = \partial D(F_t(x)) = \partial D(x(t))$. Thus the $\omega_t$-area of $D(x(t))$ is $t \cdot A(x(t)) = \int_{D(x(t))} \omega_t = \int_{F_t D(x)} \omega_t = \int_{D(x)} F_t^* \omega_t = \int_{D(x)} F_{t_0}^* \omega_{t_0} = t_0 \cdot A(x)$, and hence $A(x(t)) = \frac{t}{t_1} A(x)$, which tells us that all flow lines of $Y_t$ are decreasing in $A$. Since $A : v_1^0 \to [0, 1)$ is proper, flow lines cannot escape from $v_1^0$ as $t$ increases, and the flow is globally defined as a map $F : v_1^0 \times [t_0, t_1] \to v_1^0$.

For any $x \in v_1^0$, $A(x) < 1$, so $A(F_n(x)) = A(x(t_1)) < \frac{t}{t_1}$. Thus, we may arrange for $F_{t_1}(v_1^0)$ to lie in any preassigned neighborhood $V$ of $D_0$ by choosing $t_0$ sufficiently small. Since $F_{t_1} : (v_1^0, \omega_{t_0}) \to (v_1^0, \omega_{t_1})$ is symplectic,
we get the following result with the neighborhood $V = v_1^0$ of $D_0$: For the neighborhood $v_1^0$ of $D_0$ in $(v_1^0, \omega_t)$, there is a $t_0$ with $0 < t_0 \leq t_1$ such that, for all positive $t \leq t_0$, $(v_1^0, \omega_t)$ embeds symplectically in $v_1^0$ rel $D_0$. From the above fact and Theorem 2.1, we can get a symplectic embedding $\hat{f} : (v_1^0, \omega_t) \to (M_1, \omega_{M_1})$ with $\hat{f} \circ i_0 = j_1$, for any fixed $t \in (0, t_0]$ with $t_0$ suitably small, and $\hat{f}$ is isotopic rel $D_0$ to $f$.

We would like to find a similar map from a neighborhood of $D_\infty$ in $(S, \omega_t)$ into a neighborhood of $j_2(D^2)$ in $M_2$. By construction, $v_2^0 = S - D_0$ canonically identifies the normal bundles $v_3$ and $v_0$ of $D_\infty$ and $D_0$ (reversing fiber-orientation). We also have isomorphisms $f_* : v_0 \to v_1$ and $\psi : v_1 \to v_2$ (the latter reversing orientation). Let $\psi'' : v_\infty \to v_2$ denote the composite of these (which preserves orientation). Then there is a smooth embedding $g : S - D_0 \to M_2$ (independent of $t$) with $g \circ i_\infty = j_2$ and $g_* = \psi''$ on $v_\infty$. Clearly, $M = M_1 \psi M_2$ could be constructed as a smooth manifold by composing $f^{-1}$ and $g$. However, we cannot perturb $g$ to be symplectic, since we have $i_\infty^* \omega_t = \omega_{D^2} + t \cdot i_\infty^* \eta$. To remedy this, we choose a smooth map $\mu : S \to S$ that radially rescales $v_1^0$, fixing a neighborhood of $D_\infty$ and collapsing a neighborhood of $D_0$ onto $D_0$. By composing $g^{-1} \circ \mu$, we may assume that $g^{-1}$ extends to a smooth map $\lambda : N \to S$ with $\lambda(N - g(S - D_0)) \subset D_0$, where $N$ is a neighborhood of $g(S - D_0)$. Let $\zeta = \lambda^* \eta$. Then $\zeta$ is a closed 2-form that vanishes on $N - g(S - D_0)$, since $i_0^* \eta = 0$. And $\zeta$ can be extended over $M_2$ as follows:

$$
\zeta = \begin{cases} 
\lambda^* \eta & \text{over } g(S - D_0) \\
0 & \text{over } M_2 - g(S - D_0).
\end{cases}
$$

$\zeta$ is determined by $g$ and $\eta$ (so it is independent of $\lambda$ and $t$) and $j_2^* \zeta = i_\infty^* \eta$. Let’s replace $\omega_{M_2}$ by $\tilde{\omega}_{M_2} = \omega_{M_2} + t \cdot \zeta$. Since nondegeneracy is an open condition, $\tilde{\omega}_{M_2}$ will be symplectic on $M_2$ provided that $0 < t \leq t_0$ for $t_0$ sufficiently small. Furthermore, $g|_{D_\infty} : (D_\infty, \omega_t) \to (M_2, \tilde{\omega}_{M_2})$ is a symplectic embedding. Hence we can get the same result as Theorem 2.1 for the smooth embedding $g$, and by this result, there is a compactly supported isotopy rel $D_\infty$ from $g$ to $\tilde{g} : (S - D_0, \omega_t) \to (M_2, \tilde{\omega}_{M_2})$ which is symplectic on a neighborhood $U_\infty$ of $D_\infty$.

Now we perform the symplectic summation. Let $W = \tilde{g}(U_\infty - D_\infty)$ be a neighborhood of one end of the open manifold $(M_2 - S_2) - j_2(D^2)$, where $S_2$ are...
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the interior surfaces of $S_i$ on $(T'_i)^i$ with the boundaries $S_i^1 = j_i(\partial D^2)$ $(i = 1, 2)$. The map $\tilde{g}^{-1} : (W, \tilde{\omega}_{M_2}) \rightarrow (\nu_1^0, \omega_i)$ symplectically identifies the ends of $((M_2 - \tilde{S}_2) - j_2(D^2), \tilde{\omega}_{M_2})$ and $(\nu_1^0, \omega_i)$. Let $K = \hat{f}(\nu_1^0 - U_\infty)$ and let $\varphi$ be the inverse of the symplectic embedding $\hat{f} \circ \tilde{g}^{-1} : (W, \tilde{\omega}_{M_2}) \rightarrow (M_1, \omega_{M_1})$. We use $\varphi$ to glue together the two ends of $((M_1 - \tilde{S}_1) - K, \omega_{M_1})$ and $((M_2 - \tilde{S}_2) - j_2(D^2), \tilde{\omega}_{M_2})$. The resulting symplectic manifold is diffeomorphic to $M$. As in [2], we can get a unique isotopy class of symplectic forms on $M$ as follows:

$$\omega_M = \begin{cases} \omega_{M_1} & \text{on } M_1 - \nu_1^0 \\ \{(1 - s)\omega_{M_1} + s \cdot \pi^*\omega_{D^2} \mid 0 \leq s < 1\} & \text{on } cl(\nu_1^0) \\ \{\tilde{\omega}_{M_2} = \omega_{M_2} + t \cdot \xi \mid 0 < t \leq t_0\} & \text{on } M_2 - j_2(D^2). \end{cases}$$

**Theorem 2.2.** In the above notation, we have the following results:

1. The symplectic sum $(M, \omega_M)$ is a smooth symplectic four-manifold with symplectic structures $\omega_M$.
2. $T'_2 = (T'_1)^1_\#(T'_1)^2$ is a non-exact Lagrangian surface of genus 2 in $M$ with respect to $\omega_M$.
3. $[\omega_M] \neq 0$ in $H^2(M, T'_2; \mathbb{R})$.

(2) and (3) will be shown in Lemma 4.1.)

**Lemma 2.3.** $M = M_1 \#_\varphi M_2 \cong ((M_1 - \tilde{S}_1) - K) \cup_\varphi ((M_2 - \tilde{S}_2) - j_2(D^2)) \cong S^3 \times \mathbb{R}$, where $\tilde{S}_i$ are the interior surfaces of $S_i$ on $(T'_i)^i$ with the boundaries $S_i^1 = j_i(\partial D^2)$ $(i = 1, 2)$. Hence $H^1(T'_2; \mathbb{R}) \cong H^2(M, T'_2; \mathbb{R})$ is an isomorphism, where $T'_2$ is a Lagrangian surface of genus 2 in $M$.

**Proof.** We know that $M \cong ((M_1 - \tilde{S}_1) - K) \cup_\varphi ((M_2 - \tilde{S}_2) - j_2(D^2)) \cong S^3 \times (-\infty, 0) \cup_\varphi S^3 \times (0, \infty) \cong S^3 \times (-\infty, 0] \cup_\varphi S^3 \times [0, \infty)$. Since $\varphi = (\hat{f} \circ \tilde{g}^{-1})^{-1} = \tilde{g} \circ \hat{f}^{-1}$ glues together the two ends of $((M_1 - \tilde{S}_1) - K, \omega_{M_1})$ and $((M_2 - \tilde{S}_2) - j_2(D^2), \tilde{\omega}_{M_2})$, $M \cong S^3 \times \mathbb{R}$. 

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3. The construction of an exotic symplectic form

In this section we would like to construct symplectic forms on \( S^3 \times \mathbb{R} \) from exotic symplectic forms on two smooth symplectic manifolds \( M_i (i = 1, 2) \) diffeomorphic to \( \mathbb{R}^4 \). In section 4 we will prove that the symplectic forms are exotic.

Let \( \psi \in \Omega^1(\mathbb{R}^3) \) be such that the pull-back of \( \psi \) to the torus vanishes and \( d\psi \neq 0 \), and let \( \chi \in \Omega^1(\mathbb{R}^3) \) be such that \( \chi \wedge d\psi \) is a volume on \( \mathbb{R}^3 \). Let \( \rho = \psi + x^4 \cdot \chi \in \Omega^1(\mathbb{R}^4) \). We define \( \tau \) to be the smooth one-form on \( \mathbb{R}^4 \) given by

\[
\tau = r^2 \cos r^2 d\theta + s^2 \cos s^2 d\phi,
\]

where \( \mathbb{R}^4 \) may be thought of as \( \mathbb{R}^2 \times \mathbb{R}^2 \) and \((r, \theta), (s, \phi)\) are polar coordinates on each factor.

For details, we take \( \psi = (p^{-1})^*i^*\tau \), \( \chi = (p^{-1})^*i^*\xi \), and \( \xi = *(d\tau \wedge d\phi^2) \), where \( S^3 \) is a three sphere defined by \( r^2 + s^2 = \phi^2 \), \( i: S^3 \to \mathbb{R}^4 \) the standard embedding, and \( p: S^3 - \{x\} \to \mathbb{R}^3 \) the stereographic projection, where \( x \) is a point in \( S^3 - T_1 \). Then there is an open ball \( B \) in \( \mathbb{R}^3 \) containing \( p(T_1) \) and an interval \( I \) about \( x^4 = 0 \) so that \( \omega'_{M'} (= d\rho) \) is a symplectic form on a smooth symplectic four-manifold \( M' (\cong B \times I) \) diffeomorphic to \( \mathbb{R}^4 \).

We see that \( \tau \) vanishes only on the torus \( T_1 \) defined by \( x_1^2 + x_2^2 = r^2 = \pi/2 \) and \( y_1^2 + y_2^2 = s^2 = \pi/2 \). \( T_1 \) is an exact Lagrangian torus in \( M' \), since \( \rho |_{T_1} = 0 \) and \( \omega'_{M'} |_{T_1} = d\rho |_{T_1} = 0 \). By an isomorphism \( H^1(T_1; \mathbb{R}) \cong H^2(B \times I, T_1; \mathbb{R}) \), the relative class \([\omega'_{M'}]\) vanishes in \( H^2(B \times I, T_1; \mathbb{R}) \). We call this structure \( \omega'_{M'} \) an exotic symplectic structure on \( M' \). By the same procedure as in the section 2 with \( h: (D^2, \partial D^2) \to (\mathbb{R}^4, T_1) \) defined by \( h(x_1, y_1) = (\frac{x_1}{ \sqrt{2}}, \frac{y_1}{ \sqrt{2}}, \frac{y_2}{ \sqrt{2}}, -\frac{x_2}{ \sqrt{2}}) \), we can get a unique isotopy class of symplectic forms on \( M = M_1 \sharp_{\psi} M_2 \), where \( \omega'_{M_i} = d\rho_i \) are exotic symplectic forms on \( M_i \) as follows:

\[
\omega'_{M} = \begin{cases} 
\omega'_{M_1} = d\rho_1 & \text{on } M_1 - v_1^0 \\
(1 - s)\omega'_{M_1} + s \cdot \pi^* \omega_{D^2} & |0 \leq s < 1| & \text{on } cl(v_1^0) \\
\{\tilde{\omega}'_{M_2} = \omega'_{M_2} + t \cdot \xi & |0 < t \leq t_0| & \text{on } M_2 - j_2(D^2). 
\end{cases}
\]

**THEOREM 3.1.** In the above notations, we have the following results:

1. The symplectic sum \((M, \omega'_{M})\) is a smooth symplectic four-manifold with symplectic structures \( \omega'_{M} \).
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(2) $T_2 = T_1^1 \# T_1^2$ is an exact Lagrangian surface of genus 2 in $M$ with respect to $\omega_M$.

(3) $[\omega_M] = 0$ in $H^2(M, T_2; \mathbb{R})$.

((2) and (3) will be shown in Lemma 4.2.)

We also have the following Lemma 3.2 which is similar to Lemma 2.3.

**Lemma 3.2.** $H^1(T_2; \mathbb{R}) \cong H^2(M, T_2; \mathbb{R})$ is an isomorphism, where $T_2$ is a Lagrangian surface of genus 2 in $M$.

## 4. Exotic symplectic structures

Let $(M, \omega_M)$ be the smooth symplectic four-manifold in Theorem 2.2. (1).

**Lemma 4.1.** The symplectic forms $\omega_M$ admit a non-exact Lagrangian surface $T_2'$ of genus 2 in $M$ and hence $[\omega_M] \neq 0$ in $H^2(M, T_2'; \mathbb{R})$.

**Proof.** Let $S_i^1 = j_i(\partial D^2) \cap (T_1^i)^j$ ($i = 1, 2$). Let’s divide the surface $T_2'$ into 3 parts $[T_2' \cap (M_1 - v_1^0)] \cup [T_2' \cap cl(v_1^0)] \cup [T_2' \cap (M_2 - j_2(D^2))]$. In the first part, $\omega_{M_1} |_{T_2' \cap (M_1 - v_1^0)} = 0$, since $T_2' \cap (M_1 - v_1^0) \subset (T_1^1)^{1}$ and $\omega_{M_1}|_{(T_1^1)^j} = 0$. In the second part, $\omega_{M_1} |_{T_2' \cap cl(v_1^0)} = 0$, since $T_2' \cap cl(v_1^0) = S_1^1 \subset (T_1^1)^{1}$. And $\pi^* \omega_{D^2} |_{T_2' \cap cl(v_1^0)} = \omega_{D^2} |_{S_1^1} = 0$. Thus $(1 - s)\omega_{M_1} + s \pi^* \omega_{D^2} |_{T_2' \cap cl(v_1^0)} = 0$ (0 $\leq$ $s < 1$). In the third part, $\omega_{M_2} |_{T_2' \cap (M_2 - j_2(D^2)) = 0}$, since $T_2' \cap (M_2 - j_2(D^2)) \subset (T_1^1)^{2}$ and $\omega_{M_2} |_{(T_1^1)^2} = 0$. Also $\zeta$ is zero on $T_2' \cap (M_2 - j_2(D^2))$, since $\zeta$ is zero on $M_2 - g(S - D_0)$ and $T_2' \cap (M_2 - j_2(D^2)) \subset M_2 - g(S - D_0)$. Thus $\tilde{\omega}_{M_2} |_{T_2' \cap (M_2 - j_2(D^2))} = 0$ and hence, $T_2'$ is a Lagrangian surface of genus 2 in $(M = M_1 \# M_2, \omega_M)$.

Let’s examine the exactness of the Lagrangian surface $T_2'$ in $M$. $\varphi_s'(\sum_{i=1}^{2} x_i dy_i) |_{(T_1^i)^j} = \sum_{i=1}^{2} x_i dy_i |_{T_2'} = j^*(\sum_{i=1}^{2} x_i dy_i)$ can be locally written by $\frac{\pi}{2} (\sin \theta \cos \phi - \cos \theta \sin \phi) d\phi$. Let $S_0$ be a meridian in the torus $T_1^i$ with $\theta = 0$. Then we have

$$\int_{S_0} j^* \left( \sum_{i=1}^{2} x_i dy_i \right) = -\frac{\pi}{2} \int_{0}^{2\pi} \sin \phi d\phi$$

$$= \frac{\pi}{2} \cdot 4[\cos \phi]_{0}^{\pi}$$

$$\neq 0.$$
4.1, we can easily see that \(\omega\) with \(i\) and \(T_i\) with symplectic forms admitting exact Lagrangian tori \(M_i\) are symplectic. Since \(\omega\) is not exact. Thus \(T_2\) is a non-exact Lagrangian in \(M\). By the isomorphism in Lemma 2.3, \([\omega_M] \neq 0\) in \(H^2(M, T_2^2; \mathbb{R})\).

Let \((M, \omega'_M)\) be the smooth symplectic four-manifold in Theorem 3.1.(1).

**Lemma 4.2.** The symplectic forms \(\omega'_M\) admit an exact Lagrangian surface \(T_2\) of genus 2 in \(M\) and hence \([\omega'_M] = 0\) in \(H^2(M, T_2^2; \mathbb{R})\).

**Proof.** By the same method shown in the first part of the proof of Lemma 4.1, we can easily see that \(\omega'_M|_{T_2} = 0\) and hence \(T_2\) is also a Lagrangian surface of genus 2 in \((M = M_1 \sharp \psi M_2, \omega'_M)\).

Let’s examine the exactness of the Lagrangian surface \(T_2\) in \(M\). \(\rho_1|_{T_2 \cap (M_1 - v_0)} = 0\), since \(T_2 \cap (M_1 - v_0) \subseteq T_1^1\) and \(\rho_1|_{T_1^1} = 0\). Moreover \(\pi^*(x_1 dy_1)|_{T_2 \cap (v_0^1)} = x_1 dy_1|_{v_0^1}\) is an exact form. Therefore \((1 - s) \rho_1 + s \cdot \pi^*(x_1 dy_1)|_{T_2 \cap (v_0^1)}\) is exact. We know that \(\omega_{M_1}|_{T_2 \cap (M_2 - j_2(D^2))} = d\rho_2|_{T_2 \cap (M_2 - j_2(D^2))}\), since \(\zeta\) is zero on \(T_2 \cap (M_2 - j_2(D^2)) \subseteq M_2 - g(S - D_0)\) and that \(\rho_2|_{T_1^1 \cap (M_2 - j_2(D^2))} = 0\), since \(T_2 \cap (M_2 - j_2(D^2)) \subseteq T_1^2\) and \(\rho_2|_{T_1^2} = 0\). Thus \(T_2\) is an exact Lagrangian in \(M\) and we conclude Lemma 4.2 by the use of Lemma 3.2.

By the Lemmas 4.1, 4.2, we can get the following Theorem 4.3.

**Theorem 4.3.** The symplectic forms \(\omega_M\) on the symplectic sum \(M\) of two smooth symplectic four-manifolds \(M_i\) \((i = 1, 2)\) diffeomorphic to \(\mathbb{R}^4\) with symplectic forms admitting non-exact Lagrangian tori \((T_i^1)'\) \((i = 1, 2)\) admit a non-exact Lagrangian surface \(T'_2\) of genus 2 and \([\omega_M] \neq 0\) in \(H^2(M, T_2^2; \mathbb{R})\).

On the other hand, the symplectic forms \(\omega'_M\) on the symplectic sum \(M\) of two smooth symplectic four-manifolds \(M_i\) \((i = 1, 2)\) diffeomorphic to \(\mathbb{R}^4\) with symplectic forms admitting exact Lagrangian tori \(T_i^1\) \((i = 1, 2)\) admit an exact Lagrangian surface \(T_2\) of genus 2 and \([\omega'_M] = 0\) in \(H^2(M, T_2^2; \mathbb{R})\). Therefore, \((M, \omega'_M)\) does not symplectically diffeomorphic to \((M, \omega_M)\).

In addition, we can show the exoticities of \(\omega_M\) and \(\omega'_M\) for any closed 2-form (not necessarily exact) \(\eta\) on the sphere bundle \(S \cong D^2 \times S^2\) over \(D^2\) with \(i_0^\eta = 0\) and \(\eta\) restricting to a symplectic form on each fiber, since \(T'_2 \cap ((M_2 - S_j) - j_2(D^2)) \subseteq M_2 - g(S - D_0)\).
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References


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