A NONLINEAR BEAM EQUATION WITH NONLINEARITY CROSSING AN EIGENVALUE

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ABSTRACT. We investigate the existence of solutions of the nonlinear beam equation under the Dirichlet boundary condition on the interval \((-\frac{\pi}{2}, \frac{\pi}{2})\) and periodic condition on the variable \(t\), \(Lu + bu^+ - au^- = f(x, t)\), when the jumping nonlinearity crosses the first positive eigenvalue.

0. Introduction

In this paper, we investigate multiplicity of solutions \(u(x, t)\) for a piecewise linear perturbation \(- (bu^+ - au^-)\) of the beam operator \(L\) under the Dirichlet boundary condition on the interval \((-\frac{\pi}{2}, \frac{\pi}{2})\) and periodic condition on the variable \(t\),

\[
Lu + bu^+ - au^- = f(x, t) \quad \text{in} \ (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R},
\]

\[
(0.1)
\]

\[
u(\pm \frac{\pi}{2}, t) = u_{xx}(\pm \frac{\pi}{2}, t) = 0,
\]

\[
u(x, t) = u(-x, t) = u(x, -t) = u(x, t + \pi),
\]

where \(L\) denote the beam operator \(Lu = u_{tt} + u_{xxxx}\). The eigenvalues of \(L\) under the Dirichlet boundary condition and periodic condition on the variable \(t\) are given by \(\lambda_{mn} = (2n + 1)^4 - 4m^2(m, n = 0, 1, 2, \cdots)\).

Let \(Q\) be the square \((-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\frac{\pi}{2}, \frac{\pi}{2})\) and \(H\) be the Hilbert space defined by

\[
H = \{u \in L^2(Q) \mid u \text{ is even in } x \text{ and } t\}.
\]

Then equation (0.1) is represented by

\[
(0.2) \quad Lu + bu^+ - au^- = f \quad \text{in} \ H.
\]

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In [2,10], Choi and McKenna investigated multiplicity of solutions of a semilinear equation (0.2) when the nonlinearity $-bu^+$ crosses an eigenvalue $\lambda_{10}$ and the forcing term $f$ is supposed to be $1 + \varepsilon h(\|h\|)$. In [5], Choi and Jung investigated multiplicity of solutions of a semilinear equation (0.2) when the nonlinearity $-(bu^+ - au^-)$ crosses two eigenvalues $\lambda_{00}, \lambda_{10}$ and the source term $f$ is generated by $\phi_{00}$ and $\phi_{10}$, and when the nonlinearity $-(bu^+ - au^-)$ crosses an eigenvalue $\lambda_{10}$ and the source term $f$ is generated by $\phi_{00}$ and $\phi_{10}$.

Our concern is to investigate multiplicity of solutions of (0.2) when $-17 < a < -1 < b < 3$ and the source term $f$ is generated by two eigenfunctions $\phi_{00}, \phi_{10}$. In particular, we investigate multiplicity of solutions of (0.2).

In Section 1, we suppose that the nonlinearity $-(bu^+ - au^-)$ crosses the eigenvalue $\lambda_{00}$ and the source term $f$ is generated by $\phi_{00}$ and $\phi_{10}$. And we use the variational reduction method to reduce the problem from an infinite dimensional one to a finite dimensional one. Let $Q = (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\frac{\pi}{2}, \frac{\pi}{2})$ and $V$ be the subspace of $L^2(Q)$ spanned by $\phi_{00}$ and $\phi_{10}$. Let $P$ be the orthogonal projection $L^2(Q)$ onto $V$. Then the beam equation (0.1) is reduced to an equation in $V$.

In Section 2, we define a map $\Phi$ by

$$\Phi(v) = L v + P (b(v + \theta(v))^+ - a(v + \theta(v))^-), \quad v \in V$$

and we investigate the properties of the map $\Phi$ and we reveal a relation between multiplicity of solutions and source terms in equation (0.2) when $f$ belongs to the two-dimensional space $V$. We also determine the region of source terms in which (0.2) has no solution.

### 1. A variational reduction method

We consider the beam equation under the Dirichlet boundary condition on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ and periodic condition on the variable $t$

$$u_{tt} + u_{xxxx} + bu^+ - au^- = f(x, t) \quad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R},$$

$$u(\pm \frac{\pi}{2}, t) = u_{xx}(\pm \frac{\pi}{2}, t) = 0,$$

$$u(x, t) = u(-x, t) = u(x, -t) = u(x, t + \pi).$$

(1.1)

Here we suppose that $-\lambda_{41} = -17 < a < -\lambda_{00} = -1 < b < -\lambda_{10} = 3$. 
Let $L$ be the differential operator $Lu = u_{tt} + u_{xxxx}$. Then the eigenvalue problem

$$Lu = \lambda u \quad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}) \times R,$$

(1.2)

$$u(\pm \frac{\pi}{2}, t) = u_{xx}(\pm \frac{\pi}{2}, t) = 0,$$

$$u(x, t) = u(-x, t) = u(x, -t) = u(x, t + \pi)$$

has infinitely many eigenvalues $\lambda_{mn}$ and corresponding eigenfunctions $\phi_{mn} (m, n \geq 0)$ given by

$$\lambda_{mn} = (2n + 1)^4 - 4m^2,$$

$$\phi_{mn} = \cos 2mt \cos (2n + 1)x \quad (m, n = 0, 1, 2, \ldots).$$

Let $Q$ be the square $(-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\frac{\pi}{2}, \frac{\pi}{2})$ and $H$ be the Hilbert space defined by

$$H = \{u \in L^2(Q) \mid u \text{ is even in } x \text{ and } t\}.$$ 

Then the set $\{\phi_{mn} \mid m, n = 0, 1, 2, \ldots\}$ forms an orthogonal set in $H$.

Equation (1.1) is equivalent to

$$Lu + bu^+ - au^- = f \quad \text{in } H,$$

(1.3)

where we assume that $f = s_1\phi_{00} + s_2\phi_{10}$ ($s_1, s_2 \in R$).

**Theorem 1.1.** If $s_1 < 0$, then (1.3) has no solution.

**Proof.** We rewrite (1.3) as

$$(L - \lambda_{00})u + (b + \lambda_{00})u^+ - (a + \lambda_{00})u^- = s_1\phi_{00} + s_2\phi_{10} \quad \text{in } H.$$ 

Multiply across by $\phi_{00}$ and integrate over $Q$. Since $L$ is self-adjoint and $(L - \lambda_{00})\phi_{00} = 0$, $(L - \lambda_{00})u, \phi_{00}) = 0$. Thus we have

$$\int_Q \{(b + \lambda_{00})u^+ - (a + \lambda_{00})u^-\}\phi_{00} = (s_1\phi_{00} + s_2\phi_{10}, \phi_{00})$$

$$= s_1 \int_Q \phi_{00}^2$$

$$= \frac{\pi}{2} s_1.$$
We know that \((b + \lambda_{00})u^+ - (a + \lambda_{00})u^- \geq 0\) for all real valued function \(u\). Also \(\phi_{00} > 0\) in \(Q\). Therefore \(\int_Q \{(b + \lambda_{00})u^+ - (a + \lambda_{00})u^- \} \phi_{00} \geq 0\). Hence there is no solution of (1.3) if \(s_1 < 0\). □

Let \(V\) be the subspace of \(H\) spanned by \(\{\phi_{00}, \phi_{10}\}\) and \(W\) be the orthogonal complement of \(V\) in \(H\). Let \(P\) be the orthogonal projection of \(H\) onto \(V\). Then every \(u \in H\) can be written as \(u = v + w\), where \(v = Pu\) and \(w = (I - P)u\). Hence equation (1.3) is equivalent to a system

(1.4) \[ Lw + (I - P)(b(v + w)^+ - a(v + w)^-) = 0, \]
(1.5) \[ Lv + P(b(v + w)^+ - a(v + w)^-) = s_1\phi_{00} + s_2\phi_{10}. \]

**Lemma 1.2.** For a fixed \(v \in V\), (1.4) has a unique solution \(w = \theta(v)\). Furthermore, \(\theta(v)\) is Lipschitz continuous in \(v\).

**Proof.** Let \(\delta = \frac{a+b}{2}\). Rewrite (1.4) as

(1.6) \[ (-L - \delta)w = (I - P)(b(v + w)^+ - a(v + w)^-) \]

or equivalently,

\[ w = (-L - \delta)^{-1}(I - P)g_v(w), \]

where

\[ g_v(w) = b(v + w)^+ - a(v + w)^- - \delta(v + w). \]

Since

\[ |g_v(w_1) - g_v(w_2)| \leq \max\{|b - \delta|, |\delta - a|\}|w_1 - w_2|, \]

we have

\[ \|g_v(w_1) - g_v(w_2)\| \leq \max\{|b - \delta|, |\delta - a|\}\|w_1 - w_2\|. \]

The operator \((-L - \delta)^{-1}(I - P)\) is a self-adjoint compact linear map from \(W\) into itself. Its eigenvalues in \(W\) are \((-\lambda_{mn} - \delta)^{-1}\), where \(\lambda_{mn} \neq 1\). Therefore its \(L^2\) norm is \(\max\{\frac{1}{|1 - 17 - \delta|}, \frac{1}{|3 - \delta|}\}\). Since \(\max\{|b - \delta|, |\delta - a|\} < \min\{|-17 - \delta|, |3 - \delta|\}\), for fixed \(v \in V\), the right hand side of (1.6) defines a Lipschitz mapping of \(W\) into itself with Lipschitz constant \(\gamma < 1\).

By the contraction mapping principle, for each \(v \in V\), there is a unique \(w \in W\) which satisfies (1.4).
By the standard argument principle, $\theta(v)$ is Lipschitz continuous in $v$. □

By Lemma 1.2, the study of the multiplicity of solutions of (1.3) is reduced to that of an equivalent problem

$$(1.7) \quad Lv + P(b(v + \theta(v))^+ - a(v + \theta(v))^-) = s_1\phi_0 + s_2\phi_{10}$$

defined on $V$.

**Proposition.** If $v \geq 0$ or $v \leq 0$, then $\theta(v) = 0$.

**Proof.** Let $v \geq 0$. Then $\theta(v) = 0$ and equation (1.4) is reduced to

$$L0 + (I - P)(bv^+ - av^-) = 0$$

because $v^+ = v$, $v^- = 0$ and $(I - P)v = 0$. Similarly if $v \leq 0$, then $\theta(v) = 0$. □

Since $V = \text{span}\{\phi_{00}, \phi_{10}\}$ and $\phi_{00}$ is a positive eigenfunction, there exists a cone $C_1$ defined by

$$C_1 = \{v = c_1\phi_{00} + c_2\phi_{10} \mid c_1 \geq 0, |c_2| \leq c_1\}$$

so that $v \geq 0$ for all $v \in C_1$, and a cone $C_3$ defined by

$$C_3 = \{v = c_1\phi_{00} + c_2\phi_{10} \mid c_1 \leq 0, |c_2| \leq |c_1|\}$$

so that $v \leq 0$ for all $v \in C_3$. Thus $\theta(v) \equiv 0$ for $v \in C_1 \cup C_3$.

Now we set

$$C_2 = \{v = c_1\phi_{00} + c_2\phi_{10} \mid c_2 \geq 0, |c_1| \leq c_2\}$$

$$C_4 = \{v = c_1\phi_{00} + c_2\phi_{10} \mid c_2 \leq 0, |c_1| \leq |c_2|\}.$$

Then the union of $C_1$, $C_2$, $C_3$, and $C_4$ is the space $V$.

We define a map $\Phi : V \rightarrow V$ by

$$\Phi(v) = Lv + P(b(v + \theta(v))^+ - a(v + \theta(v))^-), \quad v \in V.$$ 

Then $\Phi$ is continuous on $V$ and we have the following lemma.
LEMMA 1.3. \( \Phi(cv) = c\Phi(v) \) for \( c \geq 0 \) and \( v \in V \).

Proof. Let \( c \geq 0 \). If \( v \) satisfies

\[
L\theta(v) + (I - P)(b(v + \theta(v))^+ - a(v + \theta(v))^-) = 0,
\]

then

\[
L(c\theta(v)) + (I - P)(b(cv + c\theta(v))^+ - a(cv + c\theta(v))^-) = 0
\]

and hence \( \theta(cv) = c\theta(v) \). Therefore

\[
\Phi(cv) = L(cv) + P(b(cv + \theta(cv))^+ - a(cv + \theta(cv))^-) = L(cv) + P(b(cv + c\theta(v))^+ - a(cv + c\theta(v))^-)
\]

\[
= c\Phi(v)
\]

We investigate the image of the cones \( C_1, C_3 \) under \( \Phi \).

First, we consider the image of \( C_1 \). If \( v = c_1\phi_{00} + c_2\phi_{10} \geq 0 \),

\[
\Phi(v) = Lv + P(b(v + \theta(v))^+ - a(v + \theta(v))^-)
\]

\[
= c_1\phi_{00} - 3c_2\phi_{10} + b(c_1\phi_{00} + c_2\phi_{10})
\]

\[
= (b + 1)c_1\phi_{00} + (b - 3)c_2\phi_{10}.
\]

Thus images of the rays \( c_1\phi_{00} \pm c_1\phi_{10} (c_1 \geq 0) \) are

\[
(b + 1)c_1\phi_{00} + (b - 3)c_1\phi_{10} \quad (c_1 \geq 0).
\]

Therefore \( \Phi \) maps \( C_1 \) onto the cone

\[
R_1 = \left\{ d_1\phi_{00} + d_2\phi_{10} \mid d_1 \geq 0, |d_2| \leq \frac{-b + 3}{b + 1}d_1 \right\}.
\]

Second, we consider the image of \( C_3 \). If \( v = -c_1\phi_{00} + c_2\phi_{10} \leq 0 \) \((c_1 \geq 0, |c_2| \leq c_1)\),

\[
\Phi(v) = Lv + P(b(v + \theta(v))^+ - a(v + \theta(v))^-)
\]

\[
= -c_1\phi_{00} - 3c_2\phi_{10} + a(-c_1\phi_{00} + c_2\phi_{10})
\]

\[
= (-a - 1)c_1\phi_{00} + (a - 3)c_2\phi_{10}.
\]
Thus images of the rays \( -c_1\phi_{00} \pm c_1\phi_{10} (c_1 \geq 0) \) are
\[
(-a - 1)c_1\phi_{00} \pm (a - 3)c_1\phi_{10} \quad (c_1 \geq 0).
\]
Therefore \( \Phi \) maps \( C_3 \) onto the cone
\[
R_3 = \left\{ d_1\phi_{00} + d_2\phi_{10} \mid d_1 \geq 0, |d_2| \leq \frac{a - 3}{a + 1}d_1 \right\}.
\]
We have three cases \( R_1 \subsetneq R_3, R_3 \subsetneq R_1, \) and \( R_1 = R_3 \). The relation \( R_1 \subsetneq R_3 \) holds if and only if the nonlinearity \( -(bu^+ - au^-) \) satisfies \( b > \frac{a + 3}{a - 1} \). The relation \( R_3 \subsetneq R_1 \) holds if and only if the nonlinearity \( -(bu^+ - au^-) \) satisfies \( b < \frac{a + 3}{a - 1} \). The relation \( R_1 = R_3 \) holds if and only if the nonlinearity \( -(bu^+ - au^-) \) satisfies \( b = \frac{a + 3}{a - 1} \).

2. Multiplicity results

We consider the restrictions \( \Phi|_{C_i} (1 \leq i \leq 4) \) of \( \Phi \) to the cones \( C_i \). Let \( \Phi_i = \Phi|_{C_i}, i.e., \Phi_i : C_i \rightarrow V \).

First, we consider \( \Phi_1 \). It maps \( C_1 \) onto \( R_1 \). Let \( l_1 \) be the segment defined by
\[
l_1 = \left\{ \phi_{00} + d_2\phi_{10} \mid |d_2| \leq \frac{-b + 3}{b + 1} \right\}.
\]
Then the inverse image \( \Phi_1^{-1}(l_1) \) is the segment
\[
L_1 = \left\{ \frac{1}{b + 1}(\phi_{00} + c_2\phi_{10}) \mid |c_2| \leq 1 \right\}.
\]
By Lemma 1.3, \( \Phi_1 : C_1 \rightarrow R_1 \) is bijective.

Second, we consider \( \Phi_3 \). It maps \( C_3 \) onto \( R_3 \). Let \( l_3 \) be the segment defined by
\[
l_3 = \left\{ \phi_{00} + d_2\phi_{10} \mid |d_2| \leq \frac{a - 3}{a + 1} \right\}.
\]
Then the inverse image \( \Phi_3^{-1}(l_3) \) is the segment
\[
L_3 = \left\{ \frac{1}{a + 1}(\phi_{00} + c_2\phi_{10}) \mid |c_2| \leq 1 \right\}.
\]
By Lemma 1.3, $\Phi_3 : C_3 \rightarrow R_3$ is bijective.

### 2.1 The nonlinearity $-(bu^+ - au^-)$ satisfies $b > \frac{a+3}{a-1}$

The relation $R_1 \subsetneq R_3$ holds if and only if the nonlinearity $-(bu^+ - au^-)$ satisfies $b > \frac{a+3}{a-1}$. We investigate the images of the cones $C_2, C_4$ under $\Phi$, where

$$C_2 = \{v = c_1\phi_{00} + c_2\phi_{10} \mid c_2 \geq 0, |c_1| \leq c_2\},$$

$$C_4 = \{v = c_1\phi_{00} + c_2\phi_{10} \mid c_2 \leq 0, |c_1| \leq |c_2|\}.$$

By Theorem 1.1 and Lemma 1.2, the image of $C_2$ under $\Phi$ is a cone containing

$$R_2 = \left\{d_1\phi_{00} + d_2\phi_{10} \mid d_1 \geq 0, \frac{-a+3}{a+1}d_1 \leq d_2 \leq \frac{b-3}{b+1}d_1\right\}$$

and the image of $C_4$ under $\Phi$ is a cone containing

$$R_4 = \left\{d_1\phi_{00} + d_2\phi_{10} \mid d_1 \geq 0, \frac{-b+3}{b+1}d_1 \leq d_2 \leq \frac{a-3}{a+1}d_1\right\}.$$

We consider the restrictions $\Phi_2$ and $\Phi_4$. Define the segments $l_2, l_4$ as follows:

$$l_2 = \left\{\phi_{00} + d_2\phi_{10} \mid \frac{-a+3}{a+1} \leq d_2 \leq \frac{b-3}{b+1}\right\},$$

$$l_4 = \left\{\phi_{00} + d_2\phi_{10} \mid \frac{-b+3}{b+1} \leq d_2 \leq \frac{a-3}{a+1}\right\}.$$

We investigate the inverse images $\Phi_2^{-1}(l_2)$ and $\Phi_4^{-1}(l_4)$. We note that $\Phi_i(C_i)$ contains $R_i$, for $i = 2, 4$, respectively.

**Lemma 2.1.1.** For $i = 2, 4$, let $\gamma$ be any simple path in $R_i$ with end points on $\partial R_i$, where each ray in $R_i$ (starting from the origin) intersects only one point of $\gamma$. Then the inverse image $\Phi_i^{-1}(\gamma)$ of $\gamma$ is also a simple path in $C_i$ with end points on $\partial C_i$, where any ray in $C_i$ (starting from the origin) intersects only one point of this path.
Proof. Since \( \Phi \) is continuous and \( \gamma \) is closed in \( V \), \( \Phi_i^{-1}(\gamma) \) is closed. Suppose that there is a ray (starting from the origin) in \( C_i \), which intersects two points of \( \Phi_i^{-1}(\gamma) \), say \( p \) and \( \alpha p (\alpha > 1) \). Then \( \Phi(\alpha p) = \alpha \Phi(p) \), which implies \( \Phi(p) \in \gamma \) and \( \Phi(\alpha p) \in \gamma \). This contradicts the assumption that each ray (starting from the origin) in \( C_i \) intersects only one point of \( \gamma \).

We regard a point \( p \in V \) as a radius vector in the plane \( V \). Define the argument \( \text{arg} \ p \) to be the angle from the positive \( \phi_0 \)-axis to \( p \).

We claim that \( \Phi_i^{-1}(\gamma) \) meets all the rays (starting from the origin) in \( C_i \). If not, \( \Phi_i^{-1}(\gamma) \) is disconnected in \( C_i \). Since \( \Phi_i^{-1}(\gamma) \) is closed and meets at most one point of any ray in \( C_i \), there are two points \( p_1 \) and \( p_2 \) in \( C_i \) such that \( \Phi_i^{-1}(\gamma) \) does not contain a point \( p \in C_i \) with \( \text{arg} \ p_1 < \text{arg} \ p < \text{arg} \ p_2 \). Let \( l \) be the segment with end points \( p_1 \) and \( p_2 \) then \( \Phi_i(l) \) is a path in \( R_i \), where \( \Phi_i(p_1) \) and \( \Phi_i(p_2) \) belong to \( \gamma \). Choose a point \( q \in \Phi_i(l) \) such that \( \text{arg} \ q = \beta \text{arg} \ p_1 \). Then there exist a point \( q' \) of \( \gamma \) such that \( q' = \beta q \) for some \( \beta > 0 \). Hence \( \Phi_i^{-1}(q) \) and \( \Phi_i^{-1}(q') \) are on the same ray (starting from the origin) in \( C_i \) and \( \text{arg} \ p_1 < \text{arg} \ \Phi_i^{-1}(q') < \text{arg} \ p_2 \), which is a contradiction. This completes the proof.

Lemma 2.1.1 implies that \( \Phi_i(i = 2, 4) \) is surjective. Hence we have the following theorem.

**Theorem 2.1.2.** For \( 1 \leq i \leq 4 \), the restriction \( \Phi_i \) maps \( C_i \) onto \( R_i \). Therefore, \( \Phi \) maps \( V \) onto \( R_3 \). In particular, \( \Phi_1 \) and \( \Phi_3 \) are bijective.

The above theorem also implies the following result.

**Theorem 2.1.3.** Suppose \(-17 < a < -1 < b < 3 \) and \( b > \frac{a+3}{a-1} \). Let \( f = s_1 \phi_{00} + s_2 \phi_{10} \in V \). Then we have:

1. If \( f \in \text{Int} R \), then (1.3) has exactly two solutions, one of which is positive and the other the other is negative.
2. If \( f \in \text{Int} R_2 \cup \text{Int} R_4 \), then (1.3) has a negative solution and at least one sign changing solution.
3. If \( f \in \partial R_3 \), then (1.3) has a negative solution.
4. If \( f \in R_3^c \), then (1.3) has no solution.

### 2.2 The nonlinearity \(-(bu^+ - au^-)\) satisfies \( b < \frac{a+3}{a-1} \)

The relation \( R_3 \subsetneq R_1 \) holds if and only if the nonlinearity \(-(bu^+ - au^-)\) satisfies \( b < \frac{a+3}{a-1} \). We investigate the images of the cones \( C_2, C_4 \) under \( \Phi \). By
Theorem 1.1 and Lemma 1.2, the image of $C_2$ under $\Phi$ is a cone containing

$$R_2' = \left\{ d_1\phi_{00} + d_2\phi_{10} \mid d_1 \geq 0, \frac{b-3}{b+1}d_1 \leq \frac{-a+3}{a+1}d_1 \right\}$$

and the image of $C_4$ under $\Phi$ is a cone containing

$$R_4' = \left\{ d_1\phi_{00} + d_2\phi_{10} \mid d_1 \geq 0, \frac{a-3}{a+1}d_1 \leq \frac{-b+3}{b+1}d_1 \right\}$$

We consider the restrictions $\Phi_2$ and $\Phi_4$. Define the segments $l_2'$, $l_4'$ as follows;

$$l_2' = \left\{ \phi_{00} + d_2\phi_{10} \mid \frac{b-3}{b+1} \leq d_2 \leq \frac{-a+3}{a+1} \right\}$$

$$l_4' = \left\{ \phi_{00} + d_2\phi_{10} \mid \frac{a-3}{a+1} \leq d_2 \leq \frac{-b+3}{b+1} \right\}$$

We investigate the inverse images $\Phi_2^{-1}(l_2')$ and $\Phi_4^{-1}(l_4')$. We note that $\Phi_i(C_i)(i = 2, 4)$ contains $R_i'(i = 2, 4)$.

**Lemma 2.2.1.** For $i = 2, 4$, let $\gamma'$ be any simple path in $R_i'$ with end points on $\partial R_i'$, where each ray in $R_i'$ (starting from the origin) intersects only one point of $\gamma'$. Then the inverse image $\Phi_i^{-1}(\gamma')$ of $\gamma'$ is also a simple path in $C_i$ with end points on $\partial C_i$, where any ray in $C_i$ (starting from the origin) intersects only one point of this path.

**Proof.** The proof is similar to the proof given in Lemma 2.1.1. \(\square\)

Lemma 2.2.1 implies that $\Phi_i(i = 2, 4)$ is surjective. Hence we have the following theorem.

**Theorem 2.2.2.** For $i = 2, 4$, the restriction $\Phi_i$ maps $C_i$ onto $R_i'$. And $\Phi_1$ and $\Phi_3$ are bijective. Therefore, $\Phi$ maps $V$ onto $R_1$.

Above the theorem also implies the following results.

**Theorem 2.2.3.** Suppose $-17 < a < -1 < b < 3$ and $b < \frac{a+3}{a-1}$. Let $f = s_1\phi_{00} + s_2\phi_{10} \in V$. Then we have:

1. If $f \in \text{Int} R$, then (1.3) has exactly two solutions, one of which is positive and the other the other is negative.
2. If $f \in \text{Int} R_2' \cup \text{Int} R_4'$, then (1.3) has a positive solution and at least one sign changing solution.
3. If $f \in \partial R_1$, then (1.3) has a positive solution.
4. If $f \in R_1^c$, then (1.3) has no solution.
2.3 The nonlinearity $-(bu^+ - au^-)$ satisfies $b = \frac{a+3}{a-1}$

The relation $R_1 = R_3$ holds if and only if the nonlinearity $-(bu^+ - au^-)$ satisfies $b = \frac{a+3}{a-1}$.

We considered the map $\Phi : V \rightarrow V$ defined by

$$\Phi(v) = Lv + P(bv + \theta(v))^+ - a(v + \theta(v))^-, \quad v \in V,$$

where $-17 < a < -1 < b < 3$ and $b = \frac{a+3}{a-1}$.

We investigate the images of $C_2$ and $C_4$ under $\Phi$. For fixed $v$, we define a map $\Phi_v : (-1, 3) \rightarrow V$ as follows

$$\Phi_v(b) = Lv + P(bv + w)^+ - a(v + w)^-, \quad b \in (-1, 3),$$

where $v \in V$ and $a$ are fixed.

**Lemma 2.3.1.** $\Phi_v$ is continuous at $b_0 = \frac{a+3}{a-1}$, where $-17 < a < -1 < b_0 < 3$ and $a$ is fixed.

**Proof.** Let $\delta = \frac{a+bu}{2}$ and $-1 < b < 3$. Rewrite (1.4) as

$$(2.3.1) \quad (-L - \delta)w = (I - P)(bv + w)^+ - a(v + w)^- - \delta(v + w),$$

or equivalently,

$$(2.3.2) \quad w = (-L - \delta)^{-1}(I - P)h(b, w),$$

where

$$h(b, w) = bv + w)^+ - a(v + w)^- - \delta(v + w).$$

By Lemma 1.2, (2.3.2) has a unique solution $w = \theta_b(v)$ for fixed $b$ with $-1 < b < 3$. Let $w_0 = \theta_{b_0}(v)$. Then we have

$$w - w_0 = (-L - \delta)^{-1}(I - P)[h(b, w) - h(b_0, w_0)]$$

$$= (-L - \delta)^{-1}(I - P)[h(b, w) - h(b, w_0)]$$

$$+ (-L - \delta)^{-1}(I - P)[h(b, w_0) - h(b_0, w_0)].$$

Since

$$\|h(b, w) - h(b, w_0)\| \leq \max\{|b - \delta|, |\delta - a|\}\|w - w_0\|$$
and
\[ \gamma = \max \left\{ \frac{1}{|17 - \delta|}, \frac{1}{|3 - \delta|} \right\} \max \{ |b - \delta|, |\delta - a| \} < 1, \]
we have
\[ \|w - w_0\| \leq \gamma \|w - w_0\| + \max \left\{ \frac{1}{|17 - \delta|}, \frac{1}{|3 - \delta|} \right\} \|v + w_0\| |b - b_0|. \]
Hence
\[ \|w - w_0\| \leq \max \left\{ \frac{1}{|17 - \delta|}, \frac{1}{|3 - \delta|} \right\} \|v + w_0\| |b - b_0|, \]
which shows that \( \theta_b(v) \) is continuous at \( b_0 = \frac{a + 3}{a - 1} \). Therefore \( \Phi_v(b) \) is continuous at \( b_0 = \frac{a + 3}{a - 1} \). \( \square \)

First, we investigate the images of the cones \( C_2 \) under \( \Phi \). Let \( p_1 = \phi_{00} + \frac{a + 3}{a + 1} \phi_{10} \) and \( p_2 = \phi_{00} + \frac{b - 3}{b + 1} \phi_{10} \). We fix \( a \). Define
\[ \theta = \begin{cases} \arg p_1 - \arg p_2, & \text{if } b > \frac{a + 3}{a - 1}, \\ \arg p_2 - \arg p_1, & \text{if } b < \frac{a + 3}{a - 1}. \end{cases} \]
Then \( 0 \leq \theta \leq \frac{\pi}{2} \) and
\[ \tan \theta = \frac{|-ab + a + b + 3|}{-2a - 2b + 4}. \]
When \( b \) converges to \( \frac{a + 3}{a - 1} \), \( \tan \theta \) converges to 0. Then \( \theta \) converges to 0 since \( 0 \leq \theta \leq \frac{\pi}{2} \). We note that \( \Phi_2 \) maps \( C_2 \) onto \( R_2 \) when \( b > \frac{a + 3}{a - 1} \) and that \( \Phi_2 \) maps \( C_2 \) onto \( R'_2 \) when \( b < \frac{a + 3}{a - 1} \). When \( b \) converges to \( \frac{a + 3}{a - 1} \), the angle of two lines consisting \( \partial R_2 \) or \( \partial R'_2 \) converges to 0. Since \( \Phi_2 \) is continuous at \( \frac{a + 3}{a - 1} \), \( \Phi_2 \) maps \( C_2 \) onto the ray
\[ S_2 = \left\{ d_1 \phi_{00} + d_2 \phi_{10} \mid d_1 \geq 0, \frac{b - 3}{b + 1} d_1 \right\} \]
when \( b = \frac{a + 3}{a - 1} \).
Second, we investigate the images of the cones $C_4$ under $\Phi$.
Let $q_1 = \phi_{00} + \frac{a-3}{a+1}\phi_{10}$ and $q_2 = \phi_{00} + \frac{-b+3}{b+1}\phi_{10}$. We fix $a$. Define

$$\theta' = \begin{cases} \arg q_1 - \arg q_2, & \text{if } b > \frac{a+3}{a-1}, \\ \arg q_2 - \arg q_1, & \text{if } b < \frac{a+3}{a-1}. \end{cases}$$

Then $0 \leq \theta' \leq \frac{\pi}{2}$ and

$$\tan \theta' = \frac{|-ab + a + b + 3|}{-2a - 2b + 4}.$$

When $b$ converges to $\frac{a+3}{a-1}$, $\tan \theta'$ converges to 0. Then $\theta'$ converges to 0, since $0 \leq \theta' \leq \frac{\pi}{2}$. We note that $\Phi_4$ maps $C_4$ onto $R_4$ when $b > \frac{a+3}{a-1}$ and that $\Phi_4$ maps $C_4$ onto $R_4'$ when $b < \frac{a+3}{a-1}$. When $b$ converges to $\frac{a+3}{a-1}$, the angle of two lines consisting $\partial R_4$ or $\partial R_4'$ converges to 0. Since $\Phi_4$ is continuous at $\frac{a+3}{a-1}$, $\Phi_4$ maps $C_4$ onto the ray

$$S_4 = \left\{ d_1\phi_{00} + d_2\phi_{10} \mid d_1 \geq 0, d_2 = \frac{-b + 3}{b + 1} d_1 \right\}$$

when $b = \frac{a+3}{a-1}$.

Hence we have the following theorem.

**Theorem 2.3.2.** For $i = 2, 4$, the restriction $\Phi_i$ maps $C_i$ onto $S_i$. And $\Phi_1$ and $\Phi_3$ are bijective. Therefore, $\Phi$ maps $V$ onto $R$, where $R = R_1 = R_3$.

The above theorem also implies the following result.

**Theorem 2.3.3.** Suppose $-17 < a < -1 < b < 3$ and $b = \frac{a+3}{a-1}$. Let $f = s_1\phi_{00} + s_2\phi_{10} \in V$. Then we have:

1. If $f \in \text{Int} R$, then (1.3) has exactly two solutions, one of which is positive and the other the other is negative.
2. If $f \in \partial R$, then (1.3) has a positive solution, a negative solution, and infinitely many sign changing solutions.
3. If $f \in R^c$, then (1.3) has no solution.
References


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