ON AUTOMORPHISMS OF CERTAIN UNIPOTENT SUBGROUPS OF CHEVALLEY GROUPS OF TYPE $D_l^*$

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1. Introduction

Let $P$ be a parabolic subgroup of a Chevalley group $G$ over a field $K$ and let $U$ be the unipotent radical of $P$. The automorphism group of $U$ was determined by J.A. Gibbs [6] when $P$ is especially a Borel subgroup of $G$. For the parabolic subgroup which is not a Borel subgroup, block matrices are involved and the characterization of the automorphism group of $U$ has been done case by case. When $G$ is of type $A_l$, the automorphism group of $U$ was determined by H.P. Khor [12]. The author determined the case of type $C_l$ [9].

Let $G = D_l(K)$ be the Chevalley group of type $D_l$, $l \geq 5$, over a field $K$ with $\text{char}(K) \neq 2, 3$. Then it is shown in Ree [13] that $G$ is isomorphic to the orthogonal group $P\Omega_{2l}(K)$ of a $2l$-dimensional vector space $V$ with the suitable quadratic form. So we determine the automorphism group of the unipotent subgroups $U$ of certain parabolic subgroups $P$ of $\Omega_{2l}(K)$ instead of $G = P\Omega_{2l}(K)$ since $G$ is an image of $\Omega_{2l}(K)$ by a homomorphism which maps $U$ isomorphically: every automorphism of $U$ can be written as a product of an inner, a diagonal, a field, a central and an extremal automorphism. In this paper a parabolic subgroup $P$ of $\Omega_{2l}(K)$ is defined as the stabilizer of a flag $(V_0, V_1, \ldots, V_n)$ of isotropic subspaces of $V$ where $n \geq 4$, $\dim V_n = l$ and $\dim V_n/V_{n-1} \geq 2$. The reader may refer to the author’s doctoral thesis [9] and [11], from which we borrow notations and many results. We omit most of proofs which can be deduced without much difficulties.

In section 2 we explain some notations and preliminaries. Section 3 deals with the maximal abelian normal subgroup to make an automorphism $\sigma$ of $U$ invariant on each fundamental root subgroup modulo $U_2$. In section 4 we only
state some propositions and the main theorem which are almost identical with ones in [11]. A diagonal automorphism is used to make our automorphism $\sigma$ trivial modulo $U_2$ and some inner automorphisms are used to further trivialize $\sigma$ up to the level of height $h - 2$, where $h$ is the height of the highest root. Then we obtain the main theorem by using two more inner and extremal automorphisms and a central automorphism.

2. Notations and preliminaries

Let $M$ be an $m \times n$ matrix. Then by the $(u, v)$ entry of $M$ we mean the $(m - u + 1, n - v + 1)$ entry of $M$. That is, the number $u$ between 1 and $m$ is the $u$-th number counted from the last number $m$.

Let $V$ be a vector space of dimension $2l$ over a field $K$ of characteristic not equal to 2, 3 with $l \geq 5$ and let $f$ be the nonsingular symmetric bilinear form on $V$, which is represented by the matrix

$$J = \sum_{i=1}^{2l} e_{i,\bar{i}} = \begin{bmatrix} 0 & 1 \\ \vdots & \ddots \\ 1 & 0 \end{bmatrix}$$

with respect to a suitable basis. Then the orthogonal group $O_{2l}(K, f)$ associated with $f$ is the set of all nonsingular linear transformations $A$ of $V$ such that

$$^T AJA = J.$$ 

Let $\Omega_{2l}(K)$ be the commutator subgroup of $O_{2l}(K, f)$ and let $P$ be the parabolic subgroup of $\Omega_{2l}(K)$ which stabilizes a flag

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n \subset V$$
of isotropic subspaces of $V$ where $n \geq 4$, $\dim V_n = l$ and $\dim V_n/V_{n-1} \geq 2$. If $\dim V_n/V_{n-1} = 1$, then the elements related to super diagonal blocks cannot generate $U$ and a graph automorphism is involved. For the case of $\dim V_i/V_{i-1} = 1$ for all $i$, $P$ is the Borel subgroup and the case is done by Gibbs [6]. Then $P$ admits a Levi decomposition $P = L \cdot U$ where $L$ is the Levi subgroup of $P$ and $U$ is the unipotent radical which consists of those transformations inducing the identity map on all quotients $V_i/V_{i-1}$. Then in
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matrix notation, an element $A$ of $U$ can be written as a $2n \times 2n$ block matrix
in the form

$$I + \sum_{u<v} A_{u,v}E_{u,v}$$

where each $A_{u,v}E_{u,v}$ denotes the matrix $A_{u,v}$ in the $(u, v)$ block and $A_{u,v} \in \text{Mat}(m_u \times n_{v-1}, K)$ with $m_u = n_{u-1} = \dim V_u/V_{u-1}$. Moreover, from the condition $^T A J A = J$ the following equation

$$\sum_{u \leq k \leq v} ^T A_{k,v}J_{k,k}A_{k,u} = 0 \quad \text{unless} \quad u = k = v$$

must be satisfied, where

$$J = \sum_k J_{k,k}E_{k,k} \quad \text{and} \quad J_{k,k} = \sum_i e_{i,i}.$$  

Since each $J_{k,k}$ has the same form of just different size, we may often simply write $J$ for $J_{k,k}$ and we denote $^T J A J$ by $J A$ which is obtained by transposing $A$ about the second diagonal([/]). So $A = -^T A$ if and only if $A$ is symmetric about the second diagonal whose entries are all zeroes.

Thus $U$ contains any element of the form

$$I + A_{u,v}E_{u,v} + A_{v,u} E_{v,u} \quad \text{where} \quad A_{v,u} = -^T A_{u,v}J = -^T A_{u,v},$$

$$I + A_{u,v}E_{u,v} + A_{v,u} E_{v,u} \quad \text{where} \quad A_{v,u} = -^T A_{u,v}J = -^T A_{u,v},$$

$$I + A_{u,v}E_{u,v} \quad \text{where} \quad A_{v,u} = -^T A_{u,v}J,$$

whenever $u < v$ and $1 \leq u, v \leq n$.

Let $R_1, \ldots, R_n$ be a basis for a real $n$-dimensional Euclidean space and set

$$X_{R_u}(A_{u,u+1}) = I + A_{u,u+1}E_{u,u+1} + A_{u+1,u} E_{u+1,u},$$

$$X_{R_u}(A_{n,n}) = I + A_{n,n} E_{n,n},$$

where $1 \leq u < n$, $A_{u+1,u} = -^T A_{u,u+1}$ and $A_{n,n} = -^T A_{n,n}$. And for all $u, v$ such that $u < v$ and $1 \leq u, v \leq n$, let

$$R_{u,v} = R_u + R_{u+1} + \cdots + R_{v-1},$$

$$R_{v,u} = 2R_{v,n} + R_n \quad \text{and} \quad R_{u,v} = R_{u,v} + R_{v,u}.$$
where \( R_1 = R_{1,2}, \ldots, R_{n-1} = R_{n-1,n} \) and \( R_n = R_{n,n}. \)

Then the set \( \pi = \{ R_i \}_{1 \leq i \leq n} \) generates a (reduced) root system \( \Phi \) isomorphic to \( C_n \) with the highest root \( R_N = R_{1,1} = 2(R_1 + \cdots + R_{n-1}) + R_n. \) Thus we have an analogous notion of the height of a root in the positive root system \( \Phi^+ \) which is the set of all the above \( R_{u,v}, R_{v,u} \) and \( R_{u,v} \) for \( u < v \) and \( 1 \leq u, v \leq n. \) And \( \text{ht}(R_N) = h = 2 (= 2n - 1). \)

Also we can associate each root \( R = R_{i,j} \) in \( \Phi^+ \) with a root subgroup \( X_R = X_{i,j} \) as follows:

\[
X_{u,v} = \{ X_{u,v}(A) = I + AE_{u,v} - J AE_{v,u} | A \in \text{Mat}(R, K) \} \\
X_{u,u} = \{ X_{u,u}(A) = I + AE_{u,u} - J AE_{v,u} | A \in \text{Mat}(R, K) \} \\
X_{v,v} = \{ X_{v,v}(A) + I + AE_{v,v} | A \in \text{Mat}(R, K) \},
\]

where \( \text{Mat}(R, K) = \text{Mat}(m_u \times m_v, K) \) for \( R = R_{u,v} \) and \( R_{u,u}. \) And

\[
\text{Mat}(R_{v,v}, K) = \{ A \in \text{Mat}(m_v \times m_v) | A = -J A \}.
\]

Then we have the following commutator relations: \([a; b] = a^{-1}b^{-1}ab,\) \([a; b; c] = [[a; b]; c] \) and \([a; b]^{-1} = [b; a] \) for any \( a, b, c \in U. \) If \( R + S \notin \Phi^+, \) then \([X_R; X_S] = 1. \) Otherwise, for \( 1 \leq u < v \leq n \) and for \( 1 \leq k \leq t \leq n, \) the products can be written as,

\[
[X_{u,v}(A); X_{k,t}(B)] = X_{u,t}(AB) \quad \text{if} \ v = k, \\
[X_{u,v}(A); X_{k,t}(B)] = X_{u,t}(AB)X_{u,u}(-AB^J A) \quad \text{if} \ v = k = t, \\
= X_{u,u}(AB) \quad \text{if} \ v = k \neq t, \\
= X_{u,u}(-A^J B + B^J A) \quad \text{if} \ u = k \text{ and } v = t, \\
= X_{u,u}(-A^J B) \quad \text{if} \ u < k \text{ and } v = t, \\
= X_{k,u}(B^J A) \quad \text{if} \ u > k \text{ and } v = t,
\]

and \( X_R(A)X_R(B) = X_R(A + B) \) for all \( R \in \Phi^+. \)

For \( 1 \leq k \leq 2, \) let \( U_k \) denote the subgroup of \( U \) generated by all root subgroups of \( U \) corresponding to the roots of height at least \( k. \) Then

\[
U_k = \prod_{R \in \Phi^+} X_R(A) \mid A = 0 \text{ whenever } \text{ht}(R) < k \text{ for } A \in \text{Mat}(R, K)
\]
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and $U_k$ is a normal subgroup of $U$, for $[U; U_k] \subset U_{k+1} \subset U_k$ by the commutator relations.

Now using the commutator relations we can show that $U$ is generated by the root subgroups $X_R$ for $R \in \Phi^+$ and we have the following expected proposition and corollary.

**Proposition 2.1.** $U$ is generated by the fundamental root subgroups $X_u = X_{R_u}$ for $R_u \in \pi$.

**Proof.** Since $U$ is generated by the root subgroups $X_R$ for $R \in \Phi^+$, it’s enough to show that;

if $X_R(A) \in U$, where $ht(R) = k$, $2 \leq k \leq 2k = h$, then

$$X_R(A) = \prod_{s=1}^{k} [X_u(A_s); X_R(B_s)]$$

for some positive integer $k$ and $1 \leq u \leq n$, where $ht(R') = k - 1$.

(1) By our assumption $\dim V_n/V_{n-1} \geq 2$, we can generate the root subgroup $X_{n-1,n}$ by fundamental roots as follows:

Let $A = \sum_{r,s} a_{rs}e_{rs} \in \text{Mat}(R_{n-1,n}, K)$. Then take

$$A_s = \sum_r a_{rs}e_{rt} \in \text{Mat}(R_{n-1,n}, K),$$

$$B_s = e_{ts} - e_{\underline{2s}} \in \text{Mat}(R_{n-1,n}, K),$$

where if $s = 1$, then $t = 1$, otherwise $t = 1$. Then we have

$$A_sB_s = \sum_r a_{rs}e_{rs}, \quad \sum_s A_sB_s = \sum_{r,s} a_{rs}e_{rs} = A$$

and $A_sB_s^J A_s = 0$.

Therefore

$$\prod_s [X_{n-1,n}(A_s); X_{n-1,n}(B_s)] = \prod_s X_{n-1,n}(A_sB_s)X_{n-1,n-1}(-A_sB_s^J A_s)$$

$$= X_{n-1,n}(\sum_s A_sB_s)$$

$$= X_{n-1,n}(A).$$
(2) If \( R = R_{u,u+1} \) for \( 1 \leq u \leq n - 2 \) and \( A \in \text{Mat}(R, K) \), then there exist \( A_s \in \text{Mat}(R_{u+1}, K) \) and \( B_s \in \text{Mat}(R_{u,u+2}, K) \) such that \( A = \sum_s B_s^J A_s \) by proposition 1.2 in [12]. So we have

\[
\sum_s [X_{u+1}(A_s); X_{u,u+2}(B_s)] = \sum_s X_{u,u+1}(B_s^J A_s) = X_{u,u+1}(A).
\]

(3) If \( R = R_{u,u} \) for \( 1 \leq u < n \) and \( A \in \text{Mat}(R, K) \), then we can choose suitable \( A_s, B_s \) so that \( A = \sum A_s B_s \). So we have

\[
\prod_s [X_u(-\frac{1}{2} A_s); X_{u,u+1}(J B_s)] = \prod_s X_{u,u}(-\frac{1}{2} A_s B_s - \frac{1}{2} J B_s^J A_s)
\]

\[
= X_{u,u}(-\frac{1}{2} \sum_s A_s B_s - \frac{1}{2} \sum_s J B_s^J A_s)
\]

\[
= X_{u,u}(-\frac{1}{2} A - \frac{1}{2} J A)
\]

\[
= X_{u,u}(A).
\]

(4) The rest of root subgroups work out in an exactly same way as ones in [11].

**Corollary 2.2.** A series \( U = U_1 \supset U_2 \supset \cdots U_1 = 1 \) is the lower central series of \( U \).

**Proof.** The property \( U_k \subset [U; U_{k-1}] \) is obtained from the proof of the above proposition 2.1., and \( [U; U_{k-1}] \subset U_k \) is from commutator products.

**3. The unique maximal abelian normal subgroup of \( U \)**

In this section we use the unique maximal abelian normal subgroup to make an automorphism \( \sigma \) of \( U \) invariant on each fundamental root subgroup \( X_i \) mod \( U_2 \). Even though the contents of propositions are same as those in [11], the proofs are quite different and more refined.

**Proposition 3.1.** Let \( M_n \) be the subgroup of \( U \) generated by all the root subgroups \( X_{u,u} \) and \( X_v,v \) for \( 1 \leq u, v \leq n, u < v \). Then \( M_n \) is the unique maximal abelian normal subgroup of \( U \).
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**Proof.** By the commutator relations, it’s clear that $M_n$ is an abelian normal subgroup of $U$. Let $N$ be any maximal abelian normal subgroup of $U$. Suppose $M_n$ is properly contained in $N$, that is, $M_n \subset N$ and $M_n \neq N$. Then $N$ always has an element $B = X_{1,n}(B_{1,n}) \bmod M_n$ for some nonzero $B_{1,n} \in \text{Mat}(R_{1,n}, K)$ [11]. Therefore, $N$ contains

$$[B; X_{2,n}(V)] = X_{1,n}(-B^TV)$$

which is not the identity for some $V \in \text{Mat}(R_{2,n}, K)$. Hence $N$ is not abelian and this is a contradiction. Therefore we must have $N = M_n$ and $M_n$ is the unique maximal abelian normal subgroup of $U$.

**Corollary 3.2.** Let $\sigma$ be an automorphism of $U$. Then $\sigma$ induces an automorphism on the quotient group $U/M_n$, which is isomorphic to the unipotent group of type $A_{n-1}$ in [12].

**Proof.** It is clear because $\sigma(M_n) = M_n$.

**Proposition 3.3.** Let $\sigma$ be an automorphism of $U$. Then $\sigma$ is invariant on each fundamental root subgroup $X_i \bmod U_2$, i.e., $\sigma(X_i(A_i)) = X_i(\sigma_i(A_i)) \bmod U_2$.

**Proof.** The case of $i = n$ is given by the above proposition 3.1 since $\sigma$ is invariant on $M_n$. Let $A_i \in \text{Mat}(R_i, K)$ be nonzero for $i < n$. Then we have

$$\sigma(X_i(A_i)) = X_i(\sigma_i(A_i)) \bmod U_2M_n$$

by the above corollary, proposition 6.1 in [12] and proposition 3.3 in [11]. So we can assume

$$\sigma(X_i(A_i)) = X_i(B_i)X_n(B_n) \bmod U_2$$

where $B_i = \sigma_i(A_i) \neq 0$ for $i < n$.

Then it suffices to show that $B_n = 0$. Suppose $B_n \neq 0$, contrarily. Since the diagonal entries of the matrix in $\text{Mat}(R_{n+1}, K)$ are zeroes, we can not use nilpotent normal subgroups of class 2 as in [11]. So we need a different approach here.
In the following commutator calculations we use properties
\[ [a; bc] = [a; c][a; b][[a; b]; c] \quad \text{and} \quad [ab; c] = [a; c][[a; c]; b][b; c] \]
for \( a, b, c \in U \).

In the case of \( i \leq n - 3 \), since \( R_i + R_{n-1} \) is not a root, we have \( [X_i(A_i); X_{n-1}(A_{n-1})] = 1 \) for all \( A_{n-1} \in \text{Mat}(R_{n-1}, K) \), whence its image under \( \sigma \) should be the identity. But we have instead
\[
\sigma \left( [X_i(A_i); X_{n-1}(A_{n-1})] \right) \\
= [X_i(B_i)X_n(B_n) \mod U_2; X_{n-1}(C_{n-1})X_n(C_n) \mod U_2] \\
= X_{n-1}(B_n)(-C_{n-1}B_n) \mod U_3,
\]
which is not the identity, since \( C_{n-1}B_n \neq 0 \) for some \( \sigma(A_{n-1}) = C_{n-1} \).

In the case of \( i = n - 2 \), since \( 2R_{n-2} + R_{n-1} \) is not a root, we have
\[
\sigma \left( [X_{n-2}(A_{n-2}); X_{n-1}(A_{n-1}); X_{n-2}(A_{n-2})] \right) \\
= X_{n-2}(B_{n-2})X_n(B_n); X_{n-1}(C_{n-1})X_n(C_n); \\
= X_{n-2}(B_{n-2})X_n(B_n) \mod U_4 \\
= X_{n-2}(2B_{n-2}C_{n-1}B_n) \mod U_4,
\]
where \( 2B_{n-2}C_{n-1}B_n \neq 0 \) for some \( \sigma(A_{n-1}) = C_{n-1} \) since \( \text{char}(K) \neq 2 \).

Finally for the case of \( i = n - 1 \), refer to proposition 3.3 in [11]. Even though we can use the property that \( R_{n-2} + 2R_{n-1} \) is not a root, the proof is different from the above cases in order to make the image under \( \sigma \) being the identity.

4. The elementary automorphisms of \( U \)

Now we are ready to characterize an automorphism of \( U \) using the elementary automorphisms. For simplectic groups, all the definitions of elementary automorphisms which are necessary for the characterization of an automorphism of \( U \) are explained in section 2 of [11]. In our case of orthogonal groups, we can use the same definitions in [11] with some obvious changes. Since we assumed \( \dim V_n / V_{n-1} \geq 2 \) so that the fundamental root subgroups related to super diagonal blocks generate \( U \), we obtain the following propositions step by step. The proofs are tedious but not so difficult. Please refer to [11] for definitions of elementary automorphisms and proofs of the following propositions.
PROPOSITION 4.1. Let $\sigma$ be an automorphism of $U$ which is invariant on each fundamental root subgroup. Then there are a diagonal automorphism $d$ and a field automorphism $f$ such that $f^{-1} \cdot d^{-1} \cdot \sigma$ acts trivially on $U \mod U_2$.

Here a map $\sigma$ acts trivially on $U \mod U_2$ means $\sigma$ is the identity map on $U \mod U_2$.

PROPOSITION 4.2. Let $\sigma$ act trivially on $U \mod U_2$. Then there exists an inner automorphism $i_1$ such that $i_1^{-1} \cdot \sigma$ acts trivially on $U \mod U_{h-2}$.

PROPOSITION 4.3. Let $\sigma$ acts trivially on $U \mod U_{h-2}$. Then there exist an inner automorphism $i_2$ and an extremal automorphism $e_1$ of $U$ such that $e_1^{-1} \cdot i_2^{-1} \cdot \sigma$ acts trivially on $U \mod U_{h-1}$.

PROPOSITION 4.4. Let $\sigma$ act trivially on $U \mod U_{h-1}$. Then there exist an inner automorphism $i_3$ and an extremal automorphism $e_2$ of $U$ such that $e_2^{-1} \cdot i_3^{-1} \cdot \sigma$ acts trivially on $U \mod U_h$.

Therefore we obtain the following main theorem.

THEOREM 4.5. Suppose $\text{Char}(K) \neq 2, 3$ and $l \geq 5$. Let $U$ be the unipotent subgroup of the parabolic subgroup of $\Omega_{2l}(K)$ which stabilizes a flag

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n \subset V$$

of isotropic subspaces where $n \geq 4$, $\dim V_n = l$ and $\dim V_n / V_{n-1} \geq 2$. Then for an automorphism $\sigma$ of $U$ there exist diagonal $d$, field $f$, inner $i$, extremal $e$ and central $c$ automorphisms such that

$$\sigma = d \cdot f \cdot i \cdot e \cdot c$$

Proof. Let $\sigma$ be an automorphism of $U$. Then from the previous five propositions (3.3, 4.1, 4.2, 4.3 and 4.4), we have $e_1^{-1} \cdot i_3^{-1} \cdot e_2^{-1} \cdot i_2^{-1} \cdot i_1^{-1} \cdot f^{-1} \cdot d^{-1} \cdot \sigma$ as a central automorphism $c$. But since an extremal automorphism $e_1$ and inner automorphism $i_3$ commute, the theorem follows, where $e = e_1 \cdot e_2$, $i = i_1 \cdot i_2 \cdot i_3$. 

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References


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