HARMONIC BERGMAN SPACES OF THE HALF-SPACE AND THEIR SOME OPERATORS

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Abstract. On the setting of the half-space of the Euclidean n-space, we consider harmonic Bergman spaces and we also study properties of the reproducing kernel. Using covering lemma, we find some equivalent quantities. We prove that if \( \lim_{i \to \infty} \frac{\mu K_{r_i}(z_i)}{V K_{r_i}(z_i)} = 0 \) then the inclusion function \( I : b^p \to L^p(H_n, d\mu) \) is a compact operator. Moreover, we show that if \( f \) is a nonnegative continuous function in \( L^\infty \) and \( \lim_{z \to \infty} f(z) = 0 \), then \( T_f \) is compact if and only if \( f \in C_0(H_n) \).

1. Introduction

Let \( H_n \) be the open subset of the Euclidean space \( \mathbb{R}^n \) given by
\[
H_n = \{(x, y) : y > 0\},
\]
where we have written a typical point \( z \in \mathbb{R}^n \) as \( z = (x, y) \), with \( x \in \mathbb{R}^{n-1} \) and \( y \in \mathbb{R}^+ \), \( dV \) will be the usual n-dimensional volume measure on \( H_n \) and \( B(z, r) \) the Euclidean ball with center \( z \) and radius \( r \). For \( 1 \leq p < \infty \), let
\[
b^p = \{f \in h(H_n) : \int_{H_n} |f|^p dV < \infty\},
\]
where \( h(H_n) \) is the set of all harmonic functions on \( H_n \). Then the harmonic Bergman space \( b^p \) is a closed subspace of \( L^p(H_n, dV) \) ([2], [3], [5]).
If \( p = 2 \) then \( L^2(H_n, dV) \) is a Hilbert space and hence there is an orthogonal projection \( Q \) from \( L^2(H_n, dV) \) onto \( b^2 \). For each \( z \in H_n \), we define \( \Lambda_z : b^2 \to \mathbb{C} \) by \( \Lambda_z(f) = f(z) \) for all \( f \in b^2 \). Then \( \Lambda_z \in (b^2)^* \). Thus there exists a unique function \( R(z, \cdot) \in b^2 \) such that \( f(z) = \int_{H_n} f(w) R(z, w) dV(w) \) for all \( f \in b^2 \) and \( Q(f(z)) = \int_{H_n} f(w) R(z, w) dV(w) \). By Theorem 8.22 in [2], for \( z = (z_1, \ldots, z_n) \) and \( w = (w_1, \ldots, w_n) \in H_n \),

\[
R(z, w) = \frac{4}{nV(B)} \frac{n(z_n + w_n)^2 - |z - \overline{w}|^2}{|z - \overline{w}|^{n+2}}
\]

which is called the reproducing kernel for \( b^2 \), where \( \overline{w} = (w_1, \ldots, w_{n-1}, -w_n) \). The purpose of this paper is to study these reproducing kernels and compactness characterization for Toeplitz operators with nonnegative continuous symbols on the harmonic Bergman space of the half-plane. In Section 2, we point out how harmonic reproducing kernels behave differently from one’s on the unit disk. In Section 3, we establish some properties for \( R(z, \cdot) \) and the inclusion operator \( I : b^p \to L^p(H_n, d\mu) \), where \( \mu \) is a positive Borel measure on \( H_n \). In the last section, we give a characterization of the compactness of Toeplitz operators with nonnegative bounded symbols.

Throughout this paper, the letters \( C \) and \( C_1 \) denote some constants and we use the symbol \( \approx \) to indicate that the quotient of two quantities is bounded above and below by constants when the variables vary.

2. The reproducing kernel

Lemma 2.1. For any \( z, w \in H_n \), there is a constant \( C \) such that

\[
|R(z, w)| \leq \frac{C}{|z - \overline{w}|^n}.
\]

Proof. For any \( z = (z_1, \ldots, z_n) \) and \( w = (w_1, \ldots, w_n) \in H_n \),

\[
|R(z, w)| = \left| \frac{4}{nV(B)} \frac{n(z_n + w_n)^2 - |z - \overline{w}|^2}{|z - \overline{w}|^{n+2}} \right| \\
\leq \frac{4}{nV(B)} \frac{n|z - \overline{w}|^2 + |z - \overline{w}|^2}{|z - \overline{w}|^{n+2}} \\
= \frac{4}{nV(B)} \frac{n + 1}{|z - \overline{w}|^n}.
\]

This completes the proof. \( \square \)
Proposition 2.2. For $1 < q \leq \infty$ and $z \in H_n$, $R(z, \cdot) \in b^q$.

Proof. For $x, s \in \mathbb{R}^{n-1}$ and $y \in \mathbb{R}^+$,
\[
P_{H_n}((x, y), s) = \frac{2}{nV(B)} \frac{y}{|(x, y) - s|^n}
\]
is the Poisson kernel for $H_n$ and hence
\[
\int_{\mathbb{R}^{n-1}} \frac{2}{nV(B)} \frac{y}{(|x - s|^2 + y^2)^{n/2}} ds = 1.
\]

If $w = (s, t)$ and $z = (x, y)$ where $s, x \in \mathbb{R}^{n-1}$ and $t, y \in \mathbb{R}^+$ then
\[
\int_{H_n} |R(z, w)|^q dV(w) \\
\leq C^q \int_{H_n} \frac{1}{|z - w|^n} dV(w) \text{ by Lemma 2.1} \\
= C^q \int_0^\infty \int_{\mathbb{R}^{n-1}} \frac{1}{(|x - s|^2 + (y + t)^2)^{nq/2}} ds dt \\
\leq C^q \int_0^\infty \int_{\mathbb{R}^{n-1}} \frac{1}{(y + t)^{(n-1)+1}} \frac{(y + t)}{(|x - s|^2 + (y + t)^2)^{n/2}} ds dt \\
= C_1 \int_0^\infty \frac{1}{(y + t)^{(n-1)+1}} dt \\
= C_1 \int_y^\infty \frac{1}{t^{n-1+1}} dt < \infty.
\]

Since $R(z, \cdot)$ is harmonic, $R(z, \cdot)$ is in $b^q$. \qed

Lemma 2.3. For $1 < p < \infty$, there exists $C$ such that
\[
\int_{H_n} \frac{w_n^{-1/p}}{|z - w|^n} dV(w) = C z_n^{-1/p}
\]
for all $z \in H_n$.  

Harmonic Bergman spaces
Proof. Fix $z = (x, y) \in H_n$. Letting $w = (s, t)$ where $s \in \mathbb{R}^{n-1}$ and $t \in \mathbb{R}^+$, we have

$$
\int_{H_n} \frac{t^{-1/p}}{|z-w|^n} dV(w) = \int_0^\infty \int_{\mathbb{R}^{n-1}} \frac{t^{-1/p}}{|(x,y) - (s,-t)|^n} dsdt \\
= \int_0^\infty \frac{t^{-1/p}}{y+t} \int_{\mathbb{R}^{n-1}} \frac{y+t}{|x-y - (s,-t)|^n} dsdt \\
= \int_0^\infty \frac{t^{-1/p}}{y+t} nV(B) dt \\
= \frac{nV(B)}{2} \int_0^\infty \frac{t^{-1/p}}{y+t} dt \\
= y^{-1/p} nV(B) \int_0^\infty \frac{t^{-1/p}}{1+t} dt.
$$

Since $\int_0^\infty \frac{t^{-1/p}}{1+t} dt < \infty$, $\frac{nV(B)}{2} \int_0^\infty \frac{t^{-1/p}}{1+t} dt$ is constants and hence

$$
\int_{H_n} \frac{t^{-1/p}}{|z-w|^n} dV(w) = Cy^{-1/p} \text{ for some constants } C.
$$

Suppose that $p \in (1, \infty)$ and $f \in L^p(H_n, dV)$. We note that $R(z, \cdot)$ is harmonic. By the Lebesgue dominated convergence theorem,

$$
Q(f)(z) = \int_{H_n} f(w) R(z, w) dV(w)
$$

is harmonic. Suppose $\frac{1}{p} + \frac{1}{q} = 1$. By Lemma 2.1,

$$
|Q(f(z))| = \left| \int_{H_n} f(w) R(z, w) dV(w) \right| \\
\leq C \int_{H_n} |f(w)| \frac{1}{|z-w|^n} dV(w) \\
= C \int_{H_n} |f(w)| \frac{|w_n|^{1/p} |w_n|^{-1/p}}{|z-w|^n/p |z-w|^{n/q}} dV(w).
$$
By the Hörder’s inequality and Lemma 2.3,
\[
\int_{H_n} \left| Q(f(z)) \right|^p dV(z) 
\leq \int_{H_n} C^p \int_{H_n} \left| f(w) \right|^p \frac{1}{|z - \overline{w}|^{n/p}} |z - \overline{w}|^{n/q} dV(w)^p dV(z) 
\leq C^p \int_{H_n} \int_{H_n} |f(w)|^p \frac{w_n^{1/q}}{|z - \overline{w}|^n} dV(w) \left( \int_{H_n} \frac{w_n^{-1/p}}{|z - \overline{w}|^n} dV(w) \right)^{p/q} dV(z) 
\leq C^p C_1^{p/q} \| f \|_p^p.
\]
Thus \( Q : L^p(H_n, dV) \longrightarrow b^p \) is a bounded linear operator.

We want to find some equivalent quantities of the reproducing kernel. To do so, for any \( r \in (0, 1) \) and any \( z \in H_n \), we define \( K_r(z) = \{ w \in H_n : |w - z| < rz \} \). Then we have the following lemma([4]).

**Lemma 2.4.** For \( r \in (0, \frac{4}{5}) \), there exists a sequence \( \{ z_i \} \) in \( H_n \) such that (1) \( \cup K_r(z_i) = H_n \) and (2) there is \( M \in \mathbb{N} \) such that for each \( z \in H_n \), \( \{ i : z \in K_r(z_i) \} \leq M \).

**Proof.** Let \( w_m = (s_m, t_m) \) and \( B_m = B(w_m, \frac{1}{5} t_m) \) where \( s_m \in \mathbb{Q}^{n-1} \) and \( t_m \in \mathbb{Q}^+ \). Then \( \cup B_m = H_n \). Put \( D_1 = B_1 \). For \( n \geq 2 \), we define \( D_n = B_k \), where \( k \) is the first element of the \( \{ i : B_i \cap (\cup_{j=1}^{n-1} D_j) = \emptyset \} \) and let \( z_m = (x_m, y_m) \) denote the center of \( D_m \) where \( x_m \in \mathbb{R}^{n-1} \) and \( y_m \in \mathbb{R}^+ \). Take any \( z \in H_n \). Then \( z \in B_m \) for some \( m \). If \( B_m \cap D_l = \emptyset \) for \( l \leq m - 1 \) then \( D_m \cap B_m \) and hence \( z \in K_r(z_m) \). If \( B_m \cap D_l = \emptyset \) for some \( l \leq m - 1 \) then \( t_m - y_l \leq |t_m - y_l| \leq |w_m - z_l| < \frac{5}{9} t_m + \frac{5}{9} y_l \), i.e., \( t_m < \frac{5}{9} t_m + \frac{5}{9} y_l \). Thus \( |z - z_l| \leq |z - w_m| + |w_m - z_l| < \frac{2}{9} t_m + \frac{5}{9} y_l < \frac{5}{9} t_m + \frac{5}{9} y_l < \frac{5}{9} t_m + \frac{5}{9} y_l = r(\frac{10 + 2 r + 5}{9}) y_l < r y_l \). This implies \( z \in K_r(z_l) \).

Take any \( z = (x, y) \) in \( H_n \). Let \( N_z = \{ m : |z - z_m| < 3 r y_m \} \). For \( m \in N_z \) and \( w \in K_r(z_m) \), \( |z - w| \leq |z - z_m| + |z_m - w| < 3 r y_m + \frac{5}{9} y_m = \frac{16}{5} y_m < \frac{16}{5(1 - 3r)} y \) and hence \( K_r(z_m) \leq K_{\frac{16}{5}(1 - 3r)}(z) \). Since \( \{ K_{r}(z_m) \} \) is disjoint and
\[
\sum_{m \in N_z} |K_r(z_m)| = C \pi \sum_{m \in N_z} \left( \frac{r}{5} y_m \right)^n 
\leq C \pi \left( \frac{r}{5} \right)^n \left( \frac{y}{5(1 - 3r)} \right)^n |N_z|, \quad |N_z| < \left( \frac{16(1 + 3r)}{1 - 3r} \right)^n.
\]
Thus \( \{ N_z : z \in H_n \} \) is uniformly bounded.

**Proposition 2.5.** For \( z \in H_n \) and \( w \in K_r(z) \), \( R(z, w) \approx \frac{1}{z_n} \).
Proof. Since \[ |R(z, w)| \leq \frac{C}{|z - w|^n} \] for all \( z, w \in H_n \), \( |R(z, w)| \leq \frac{C}{z^n} \).

Since \( n(z_n + w_n)^2 > |z - w|^2 \),

\[
|R(z, w)| = \frac{4}{nV(B)} \frac{n(z_n + w_n)^2 - |z - w|^2}{|z - w|^{n+2}} \geq \frac{C_1}{z_n^n}
\]

for some \( C_1 \) and hence \( R(z, w) \approx \frac{1}{z_n^n} \). \( \square \)

**Proposition 2.6.** For \( 1 < p < \infty \) and \( z \in H_n \),

\[
\|R(z, \cdot)\|_p \approx z_n^{-n(p-1)/p}.
\]

**Proof.** By Proposition 2.5,

\[
\|R(z, \cdot)\|_p^p = \int_{H_n} |R(z, w)|^p dV(w)
\]

\[
\geq \int_{K_r(z)} |R(z, w)|^p dV(w)
\]

\[
= \int_{K_r(z)} \frac{C}{z_n^{np}} dV(w)
\]

\[
= CC_1 \frac{z_n^n}{z_n^{np}} = CC_1 z_n^{-n(p-1)}.
\]

Note that

\[
\|R(z, \cdot)\|_p^p = \int_{H_n} |R(z, w)|^p dV(w)
\]

\[
\leq C_1 \int_{H_n} \frac{1}{|z - w|^{np}} dV(w)
\]

\[
\leq C_1 C_2 \int_0^\infty \frac{1}{(zn + wn)^{n(p-1)+1}} dw_n
\]

\[
= C_1 C_2 z_n^{-n(p-1)+1}.
\]

The proof is complete. \( \square \)
3. The embedding operator

Suppose that $1 \leq p < \infty$, $\mu$ is a positive Borel measure on $H_n$ and
\(\{K_r(z_i)\}\) is the sequence in Lemma 2.4. Let $I : b^p \to L^p(H_n, d\mu)$ be
the inclusion function. Suppose that $\frac{\mu(K_r(z_i))}{V(K_r(z_i))} < N$ for all $i = 1, 2, \ldots$
and $M$ is the multiplicity in Lemma 2.4. Then we can show that $I$ is a
function. To do so, we need the following:

**Lemma 3.1.** For $0 < r < t < 1$ and $1 \leq p < \infty$, there exists a finite
constant $C$ such that $|f(w)|^p \leq \frac{C}{V(K_t(z))} \int_{K_t(z)} |f|^p dV$
for all $z \in H_n$, $w \in K_r(z)$ and all harmonic functions $f$ on $H_n$.

**Proof.** Let $z \in H_n$ and let $w \in K_r(z)$. Since $r < t$, for any harmonic
function $f$ on $H_n$, $|f(w)|^p = \left| \frac{1}{V(B(w,(t-r)z_n))} \int_{B(w,(t-r)z_n)} f dV \right|
\leq \frac{C_1^p}{V(K_t(z))} \int_{K_t(z)} |f|^p dV$ for some constant $C_1$.

This implies the result. \qed

Take any $u$ in $b^p$. Then
\[
\int_{H_n} |u(z)|^p d\mu(z) \leq \sum_{i=1}^{\infty} \int_{K_r(z_i)} |u(z)|^p d\mu(z) 
\leq \sum_{i=1}^{\infty} \mu(K_r(z_i)) \times \sup_{z \in K_r(z_i)} |u(z)|^p 
= C \sum_{i=1}^{\infty} \frac{\mu(K_r(z_i))}{V(K_r(z_i))} \int_{K_r(z_i)} |u(z)|^p dV 
\leq CN \sum_{i=1}^{\infty} \int_{K_r(z_i)} |u(z)|^p dV(z) 
\leq CNM \int_{H_n} |u(z)|^p dV(z).
\]

Since $u \in b^p$, $I : b^p \to L^p(H_n, d\mu)$ is a function. In fact, we can show
that $I$ is compact whenever $\lim_{n \to \infty} \frac{\mu(K_r(z_i))}{V(K_r(z_i))} = 0$. 

Lemma 3.2. For $1 < p < \infty$, $b^p \cap L^\infty$ is dense in $b^p$.

Proof. Take any $\varepsilon > 0$ and any $f$ in $b^p$. For each $\delta > 0$ and any $z = (x, y)$, let $f_\delta(z) = f(x, y + \delta)$. Then $f_\delta \in b^p$. Since $C_c(H_n)$ is dense in $L^p$, there is $g \in C_c(H_n)$ such that $\|g - f\|_p < \varepsilon$. Since $\lim_{\delta \to 0} \|g_\delta - g\|_p = 0$, there is $\delta_0 > 0$ such that for $0 < \delta < \delta_0$, $\|g_\delta - g\|_p < \varepsilon$ and hence $\|f_\delta - f\|_p \leq \|f_\delta - g_\delta\|_p + \|g_\delta - g\|_p + \|g - f\|_p < 3\varepsilon$. Then for any $w = (s, t) \in H_n$,

$$|f_\delta(w)|^p = |f(s, t + \delta)|^p = \left|\frac{1}{V(B((s, t + \delta), \delta))} \int_{B((s, t + \delta), \delta)} f(z) dV(z)\right|^p \leq \frac{1}{V(B((s, t + \delta), \delta))} \int_{H_n} |f(z)|^p dV(z).$$

This implies $f_\delta \in L^\infty$. Thus $b^p \cap L^\infty$ is dense in $b^p$.

Proposition 3.3. For $1 < p < \infty$ and $z \in H_n$, $\frac{R(z, \cdot)}{\|R(z, \cdot)\|_p}$ converges weakly to 0 in $b^p$ as $z_n \to 0$.

Proof. Let $\frac{1}{p} + \frac{1}{q} = 1$. Take any $v$ in $b^p \cap L^\infty$. Since $\|R(z, \cdot)\|_p \approx z_n^{-(p-1)/p}$,

$$\left|\frac{1}{\|R(z, \cdot)\|_p} v(z)\right| = \frac{1}{\|R(z, \cdot)\|_p} |v(z)| \approx z_n^{n/q} |v(z)|$$

and hence $\frac{R(z, \cdot)}{\|R(z, \cdot)\|_p}$ converges weakly to 0 as $z_n \to 0$.

Lemma 3.4. Let $1 < p < \infty$ and let $\{f_m\}$ be a sequence in $b^p$. Then $\{f_m\}$ converges weakly to $f$ in $b^p$ if and only if $\{\|f_m\|_p : m \in \mathbb{N}\}$ is bounded and $\{f_m\}$ converges uniformly to $f$ on each compact subset of $H_n$.

Proof. Suppose $\{f_m\}$ is a sequence in $b^p$ such that $\{f_m\}$ converges weakly to $f$ in $b^p$. For each $g \in b^p$, we define $A_g : (b^p)^* \to \mathbb{C}$ by $A_g(v) = v(g)$ for all $v \in (b^p)^*$. Then $\|A_g\| = \|g\|_p$. We note that $\{f_m\}$ converges weakly to $f$ if and only if $\lim_{m \to \infty} v(f_m) = v(f)$ for all $v \in (b^p)^*$ if and only if $\{A_{f_m}\}$ converges pointwise to $f$ in $(b^p)^*$ and $(b^p)^*$ is a Banach space. By the uniform boundedness principle, $\sup\{\|A_{f_m}\| : m \in \mathbb{N}\} = \sup\{\|f_m\|_p : m \in \mathbb{N}\}$ is bounded. For any
\[ g \in b^p, \quad z \in H_n \quad \text{and any compact subset } K \text{ of } H_n, \]
\[ |g(z)|^p = \left| \frac{1}{V(B(z, z_n))} \int_{B(z, z_n)} g(w) dV(w) \right|^p \leq \frac{1}{V(B(z, z_n))} \int_{B(z, z_n)} |g(w)|^p dV(w) \leq \frac{1}{V(B(z, z_n))} \|g\|_p^p. \]

By the Arzela-Ascoli theorem, for any compact subset \( K \), there is a subsequence \( \{f_{m_k}\} \) of \( \{f_m\} \) such that \( \{f_{m_k}\} \) converges uniformly to \( f \) on \( K \). Since \( \{f_m\} \) converges pointwise to \( f \) and \( \{f_{m_k}\} \) converges uniformly to \( f \) on \( K \), \( \{f_m\} \) converges uniformly to \( f \) on each compact subset on \( H_n \).

Conversely, take any \( v \) in \( (C_c(H_n))^* \). By the Riesz representation theorem, there exists a unique regular, complex-valued Borel measure \( \mu \) such that \( v(g) = \int_{H_n} g d\mu \) for all \( g \in C_c(H_n) \). Note that \( \{f_m\} \) converges pointwise to \( f \). By the Lebesgue dominated convergence theorem, \( \lim_{n \to \infty} v(f_m) = \lim_{m \to \infty} \int_{H_n} f_m d\mu = \int_{H_n} f d\mu = v(f) \) and hence \( \{f_m\} \) converges weakly to \( f \).

**Theorem 3.5.** If \( \lim_{i \to \infty} \frac{\mu(K_r(z_i))}{V(K_r(z_i))} = 0 \) then the embedding operator \( I \) is compact.

**Proof.** Suppose \( \{f_m\} \) converges weakly to 0 in \( b^p \). By Lemma 3.4, \( \{f_m\} \) converges uniformly on each compact subset of \( H_n \) and \( \{\|f_m\|_p : m \in \mathbb{N}\} \) is bounded. Let \( \varepsilon > 0 \) be given. Since \( \lim_{i \to \infty} \frac{\mu(K_r(z_i))}{V(K_r(z_i))} = 0 \), there is \( k \in \mathbb{N} \) such that for \( i \geq k \), \( \frac{\mu(K_r(z_i))}{V(K_r(z_i))} < \varepsilon \). By Lemma 3.1, there is a constant \( C \) such that \( |f_m(z)|^p \leq \frac{C}{V(K_{3r}(z_i))} \int_{K_{3r}(z_i)} |f_m|^p dV \)
for all $z \in K_r(z_i)$. Then
\[
\int_{H_n} |f_m(z)|^p d\mu(z) \leq \sum_{i=1}^{k} \int_{K_r(z_i)} |f_m(z_i)|^p d\mu(z) + \sum_{i=k+1}^{\infty} \int_{K_r(z_i)} |f_m(z)|^p d\mu(z)
\]
\[
= \sum_{i=1}^{k} \int_{K_r(z_i)} |f_m(z)|^p d\mu(z) + \sum_{i=k+1}^{\infty} \mu(K_r(z_i)) \int_{K_{3r}(z_i)} |f_m(z)|^p dV(z)
\]
\[
\leq \sum_{i=1}^{k} \int_{K_r(z_i)} |f_m(z)|^p d\mu(z) + C \sum_{i=k+1}^{\infty} \frac{\mu(K_r(z_i))}{V(K_r(z_i))} \int_{K_{3r}(z_i)} |f_m(z)|^p dV(z)
\]
where $M$ is the constant in Lemma 2.4. Since \( \{f_m\} \) converges uniformly to 0 each compact subset of $H_n$, \[
\lim_{m \to \infty} \sum_{i=1}^{k} \int_{K_r(z_i)} |f_m(z)|^p d\mu(z) = \sum_{i=1}^{k} \int_{K_r(z_i)} \lim_{m \to \infty} |f_m(z)|^p d\mu(z) = 0
\]
and hence $\lim_{m \to \infty} \|I(f_k)\|_p = 0$. Thus $I$ is compact.

4. Toeplitz operators on harmonic Bergman spaces

We note that $Q : L^2(H_n, dV) \to b^2$ is the Bergman projection. For $f \in L^\infty$, we define $T_f : b^2 \to b^2$ by $T_f(g) = Q(fg)$ for all $g \in b^2$, which is called the Toeplitz operator with symbol $f$ ([1]). Since $\|Q\| \leq 1$, $\|T_f\| \leq \|f\|_\infty$.

**Theorem 4.1.** Suppose $0 < r < 1$, $1 \leq p < \infty$ and $\mu$ is a positive Borel measure on $H_n$. Then
\[
\sup_{f \in \mathcal{B}_p} \frac{\int_{H_n} |f|^p d\mu}{\int_{H_n} |f|^p dV} \approx \sup_{z \in H_n} \frac{\mu(K_r(z))}{V(K_r(z))}.
\]

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Proof. For \( z \in H_n \), let \( g(w) = R(z, w)^{2/p} \). Then \( \int_{H_n} |g(w)|^p dV(w) = R(z, z) \). By Proposition 2.5,

\[
\int_{H_n} |g(w)|^p d\mu(w) \geq \int_{K_r(z)} |R(z, w)|^2 d\mu(w) \approx \int_{K_r(z)} \frac{1}{z_n^2} d\mu = \frac{1}{z_n^2} \mu(K_r(z)).
\]

Since \( |R(z, w)| = C_1 \frac{1}{z_n} \) for some \( C_1 \),

\[
\frac{\int_{H_n} |g(w)|^p d\mu(w)}{\int_{H_n} |g(w)|^p dV(w)} \geq C_2 \frac{\frac{1}{z_n^2} \mu(K_r(z))}{\frac{1}{z_n^2}} = C \frac{\mu(K_r(z))}{V(K_r(z))} \text{ for some } C_2 \text{ and } C.
\]

This implies that \( \sup_{f \in b} \int_{H_n} |f|^p d\mu \geq \sup_{f \in b} C \frac{\mu(K_r(z))}{V(K_r(z))} \).

Suppose \( \{K_r(z_i)\} \) is the sequence and \( M \) is the constant in Lemma 2.4. Let \( f \in b^p \) be such that \( f \not\equiv 0 \). Then

\[
\int_{H_n} |f|^p d\mu \leq \sum_{i=1}^{\infty} \int_{K_r(z_i)} |f|^p d\mu
\]

\[
\leq \sum_{i=1}^{\infty} \sup_{w \in K_r(z_i)} |f(w)|^p \mu(K_r(z_i))
\]

\[
\leq C \sum_{i=1}^{\infty} \frac{\mu(K_r(z_i))}{V(K_{1+\varepsilon}(z_i))} \int_{K_{1+\varepsilon}(z_i)} |f|^p dV
\]

\[
\leq CM \sup_{z \in H_n} \frac{\mu(K_r(z))}{V(K_r(z))} \|f\|^p.
\]

Thus

\[
\sup_{f \in b} \frac{\int_{H_n} |f|^p d\mu}{\int_{H_n} |f|^p dV} \approx \sup_{z \in H_n} \frac{\mu(K_r(z))}{V(K_r(z))}.
\]

\[\square\]

**Proposition 4.2.** Let \( K \) be a compact subset of \( H_n \). If \( f \) is in \( L^\infty \) and \( f \equiv 0 \) on \( H_n \setminus K \) then \( T_f \) is compact.

**Proof.** Let \( \{g_m\} \) be a norm bounded sequence in \( b^2 \). Take any compact subset \( K_1 \) of \( H_n \). By Hölder’s inequality, for any \( z \in K_1 \),

\[
|g_m(z)| \leq \int_{H_n} |g_m(w) R(z, w)| dV(w) \leq \|g_m\|_2 \|R(\cdot, w)\|_2 \text{ and hence there is a harmonic function } g \text{ on } H_n \text{ and a subsequence } \{g_{m_k}\} \text{ of } \{g_m\} \text{ which converges uniformly on } K_1 \text{ to } g. \]

Since \( T_f \) is continuous, \( T_f(g_{m_k}) \) converges to \( T_f(g) \). Thus \( T_f \) is compact. \[\square\]
Proposition 4.3. Let \( f \) be a nonnegative function in \( L^\infty \). If there exists \( r \in (0, 1) \) such that \( \lim_{z_n \to 0} \frac{1}{V(K_r(z))} \int_{K_r(z)} f(w) dV(w) = 0 \), then \( T_f \) is compact.

Proof. For each \( k \in \mathbb{N} \), let \( D_k = [-k, k] \times \cdots \times [-k, k] \times [\frac{1}{k}, k] \) and let \( f_k(z) = f(z)\chi_{D_k}(z) \). Then \( D_k \) is compact. By Proposition 4.2, each \( T_f \) is compact. Then

\[
\|T_f - T_{f_k}\|^2 = \sup_{\|u\|_2 = 1} \| (T_f - T_{f_k})(u) \|^2_2 \\
= \sup_{\|u\|_2 = 1} \int_{H_n} \left| (f u - f_k u)(w)R(z, w) \right|^2 dV(w) \\
\leq \sup_{\|u\|_2 = 1} \int_{H_n \setminus D_k} \left| (f u - f_k u)(w)R(z, w) \right|^2 dV(w) \\
\leq C \sup_{\|u\|_2 = 1} \int_{H_n \setminus D_k} f^2|u|^2 dV \\
= C \sup_{\|u\|_2 = 1} \int_{H_n} \chi_{H_n \setminus D_k} f^2|u|^2 dV \\
\leq C_1 C \sup_{z \in H} \frac{\int_{H_n} \chi_{K_r(z)} \chi_{H_n \setminus D_k} f^2 dV}{V(K_r(z))} \\
\leq C_1 C \|f\|_\infty \sup_{z \in H} \frac{\int_{H_n} \chi_{K_r(z)} \chi_{H_n \setminus D_k} f^2 dV}{V(K_r(z))}.
\]

By the assumption, \( \lim_{k \to \infty} \|T_f - T_{f_k}\| = 0 \). By Proposition 4.2, each \( T_{f_k} \) is compact and hence \( T_f \) also compact.

Theorem 4.4. Let \( f \) be a nonnegative function in \( L^\infty \). Then the following are equivalent:

1. \( T_f \) is compact.
2. There exists \( r \in (0, 1) \) such that \( \lim_{z_n \to 0} \frac{1}{V(K_r(z))} \int_{K_r(z)} f(w) dV(w) = 0 \).
3. For any \( r \in (0, 1) \), \( \lim_{z_n \to 0} \frac{1}{V(K_r(z))} \int_{K_r(z)} f(w) dV(w) = 0 \).

Proof. It is clear that (3) implies (2). By Proposition 4.3, (2) implies (1). It is enough to show that (1) implies (3) to complete the proof.
Suppose $T_f$ is a compact operator and $z = (x, y)$. For any $r \in (0, 1)$,
\[
\frac{1}{V(K_r(z))} \int_{K_r(z)} f(w)dV(w)
\]
\approx \int_{K_r(z)} f(w) \frac{|R(z, w)|^2}{V(K_r(z))} dV(w)
\]
by Proposition 2.5 and Proposition 2.6
\[
= \int_{H_n} \frac{f(w)}{\|R(z, \cdot)\|_2^2} R(z, w) \int_{H_n} R(w, t) R(z, t) dV(t) dV(w)
\]
\[
\leq \int_{H_n} \frac{f(w)}{\|R(z, \cdot)\|_2^2} R(z, w) \int_{H_n} R(w, t) R(z, t) dV(t) dV(w)
\]
\[
= \int_{H_n} \int_{H_n} \frac{f(w)}{\|R(z, \cdot)\|_2^2} R(z, w) R(t, w) dV(w) \frac{R(z, t)}{\|R(z, \cdot)\|_2^2} dV(t)
\]
\[
= \int_{H_n} \left( \frac{f(R(z, \cdot))}{\|R(z, \cdot)\|_2^2} \right)(t) \frac{R(z, t)}{\|R(z, \cdot)\|_2^2} dV(t)
\]
\[
= \langle T_f \left( \frac{R(z, \cdot)}{\|R(z, \cdot)\|_2^2} \right), \frac{R(z, \cdot)}{\|R(z, \cdot)\|_2^2} \rangle.
\]
By Proposition 3.3, $\lim_{z_n \to 0} \frac{1}{V(K_r(z))} \int_{K_r(z)} f(w)dV(w) = 0$. \qed

**Theorem 4.5.** Suppose that $f$ is a nonnegative continuous function in $L^\infty$ and $\lim_{z \to \infty} f(z) = 0$. Then the following are equivalent:

1. $T_f$ is compact.
2. $\lim_{z \to \partial H_n} f(z) = 0$.
3. $f \in C_0(H_n)$.

**Proof.** Suppose $T_f$ is compact and $\lim_{z \to z_0} f(z) > 0$ for some $z_0 \in \partial H_n$. Then there is $r > 0$ such that for $|z - z_0| < r$ and $z \in H_n$, $f(z) > \frac{A}{r^2}$, where $\lim_{z \to z_0} f(z) = A$. This contradicts the fact that for any $r \in (0, 1)$,
\[
\lim_{z_n \to 0} \frac{1}{V(K_r(z))} \int_{K_r(z)} f(w)dV(w) = 0.
\]
Conversely, take any $r \in (0, 1)$ any $z = (x, y) \in H_n$. Let $z_0 = (x, 0)$. Then $z_0 \in \partial H_n$ and $\lim_{z \to z_0} f(z) = 0$ and hence for any $\varepsilon > 0$ there is $\delta > 0$
such that for $|z - z_0| < \delta$, $|f(z)| < \varepsilon$. Put $z_1 = (x, \frac{y}{2})$. Then
\[
\frac{1}{V(K_r(z_1))} \int_{K_r(z_1)} f(w) dV(w) < \varepsilon.
\]
This implies that $T_f$ is compact.

It is enough to show that (2) implies (3) to complete the proof. Since $\lim_{z \to \infty} f(z) = 0$, for any $\varepsilon > 0$, there is $M > 0$ such that for $|z| > M$, $|f(z)| < \varepsilon$. Let $K = \{(x, y) : |x| \leq M \text{ and } 0 \leq y \leq M\}$. Then $K$ is compact in $\mathbb{R}^n$.

Define $g : \mathbb{R}^n \to \mathbb{C}$ by $g(z) = \begin{cases} 0, & \text{if } z_n = 0; \\ f(z), & \text{if } z_n \neq 0. \end{cases}$

Since $\lim_{z \to \partial H_n} f(z) = 0$, $g$ is continuous on $K$ and hence there is $\delta > 0$ such that for $|z_n| < \delta$, $|g(z)| < \varepsilon$. Let $K_\delta = \{z \in K : |z_n| \geq \delta\}$. Then for any $z \in H_n \setminus K_\delta$, $|f(z)| = |g(z)| < \varepsilon$. Since $K_\delta$ is compact, $f \in C_0(H_n)$. \[\square\]

References


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