FIXED POINT ITERATIONS FOR QUASI-CONTRACTION
MAPS IN UNIFORMLY SMOOTH BANACH SPACES

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1. Introduction

Suppose $K$ is a nonempty subset of a normed linear space $E$, and $T$ is a mapping of $K$ into itself. Then $T$ is called quasi-contractive (see e.g., [8]) if there exists a constant $k \in [0, 1)$ such that,

\[ k \max \{ \|x - y\|, \|x - Tx\|, \|y - Ty\|, \|x - Ty\|, \|y - Tx\| \} \]

for all $x, y \in K$. In [19], Rhoades showed that the contractive definition (1), apart from being an obvious generalization of the well-known contraction mapping, is one of the most general contractive definitions for which Picard iterations give a unique fixed point. In [18], Rhoades examined the following two fixed point iteration schemes:

(a) The Ishikawa Iteration Process: (see e.g., [10], [18]) defined as follows: For $K$ a convex subset of a Banach space $E$ and $T$ a mapping of $K$ into itself, the sequence $\{x_n\}_{n=0}^{\infty}$ in $K$ is defined by:

\[ x_0 \in K \]
\[ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n \]
\[ y_n = (1 - \beta_n)x_n + \beta_n Tx_n, \quad n \geq 0 \]

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ satisfy:

(i) $0 \leq \alpha_n \leq \beta_n < 1, \quad n \geq 0$,

(ii) $\lim_{n \to \infty} \beta_n = 0$, and

(iii) $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$
and

(b) The Mann Iteration Process: (see e.g., [13], [18]) which is defined as follows: with \( K \) and \( T \) as in (a), the sequence \( \{x_n\}_{n=0}^{\infty} \) is defined by:

\[
\begin{align*}
x_0 & \in K \\
x_{n+1} &= (1 - c_n)x_n + c_nTx_n, \quad n \geq 0
\end{align*}
\]

where,

(i) 0 \leq c_n < 1, \ n \geq 0;
(ii) \lim_{n \to \infty} c_n = 0;
(iii) \sum_{n=1}^{\infty} c_n = \infty.

In some application condition (iii) is replaced by \( \sum_{n=1}^{\infty} c_n (1 - c_n) = \infty \).

The iteration schemes (a) and (b) have been studied by various authors for approximating solutions of several nonlinear operator equations (see e.g., [4-7], [10], [13], [15], [16], [18], [20]). Moreover, it is well known that the two schemes may exhibit different behaviours for different classes of nonlinear mappings (see e.g., [18], [20]).

In [18], Rhoades showed that most of the results of [20] for the Mann iteration process can be extended to the Ishikawa’s iteration process, hence providing a much larger class of contractive fixed point iteration procedures. He further noted that the Mann iteration process can be used to approximate fixed points of quasi-contractive mappings in Hilbert spaces. He then posed the following question which remained open for many years:

QUESTION. ([18], p.747): Can the Mann iteration process be replaced by that of Ishikawa for quasi-contractive mapping of \( K \) into itself, where \( K \) is a compact convex subset of a Hilbert space?

This question was resolved in the affirmative by one of the authors [4] in a setting much more general than Hilbert spaces. Specifically, in [4] it is proved that if \( K \subseteq L_p \) (or \( \ell_p \)), \( p \in [2, \infty) \), under appropriate conditions (see e.g., [4]), the Ishikawa iteration process converges to the fixed point of a quasi-contractive mapping of \( K \) into itself. In [6], Chidume further proved that both the Mann and Ishikawa iteration methods converge strongly, \textit{without any compactness assumption on the domain of the map}, to the unique fixed point of a quasi-contractive mapping in \( L_p \) (or \( \ell_p \)), spaces,
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$p \in [2, \infty)$, and in [7] he extended these results to $L_p$ (or $\ell_p$) spaces, $p \in (2, \infty)$. Some authors (see e.g., [15], [16]) probably unaware of these results of Chidume, have recently answered the above question in the affirmative in Hilbert spaces, proving in the process results which are special cases of the results of [4], [6] and [7].

More recently, Qihou [15] considered iterates of quasi-contractive mappings still in Hilbert spaces and proved the following theorem:

**Theorem LQ** ([15], p.302). Let $T$ be a quasi-contractive mapping in a bounded closed convex subset $K$ of a Hilbert space and let $\{x_n\}_{n=0}^{\infty}$, and $\{\beta_n\}_{n=0}^{\infty}$ satisfy:

(i) $0 \leq \beta_n \leq 1; \quad n \geq 0$

(ii) $\lim_{n \to \infty} \beta_n = 0$;

(iii) $\frac{1-k^2}{2} \leq \alpha_n \leq 1 - k^2$.

Then, for each $x_0 \in K$, the sequence of Ishikawa iterates converges to the unique fixed point of $T$.

While the existence of fixed points for quasi-contractive maps has been proved in general Banach spaces (see e.g., [8], [19]), iterative methods for approximating such fixed points have largely been confined to Hilbert spaces (see e.g., [15], [16], [18]).

It is known that among all Banach spaces, Hilbert spaces are the ones with the best geometric structures (see e.g., [12], [25]) in the sense that certain geometric structures which characterize Hilbert spaces make the solution of problems posed in such spaces relatively straightforward. Consequently, to extend solutions of such problems to general Banach spaces the Banach spaces should be characterized by relations similar to those that characterize Hilbert spaces. Several authors are now conducting worthwhile research in this direction (see e.g., [1], [2], [3], [4-7], [9], [11], [14], [17], [21], [22-25]).

Recently, Hong-Kun Xu [23] studied uniformly smooth Banach spaces with modulus of smoothness 1 (defined below). These spaces include the $L_p$ (or $\ell_p$), $W_m^p$ and $e^p$ spaces, $1 < p < \infty$. Xu [23] obtained several inequalities in this space which generalize some known $L_p$ inequalities:

It is our purpose in this paper to first establish an inequality in real uniformly smooth Banach spaces with modulus of smoothness of power type $q > 1$ that
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generalizes a well known Hilbert space inequality. Using our inequality, we shall then extend the above result of Qihou [15] on the Ishikawa iteration process from Hilbert spaces to these much more general Banach spaces. Furthermore, we shall prove that the Mann iteration process converges strongly to the unique fixed point of a quasi-contractive map in this general setting. No compactness assumption on $K$ is required in our theorems.

2. Preliminaries

We begin with the following definitions: Let $E$ be a Banach space. The modulus of smoothness of $E$ is the function

$$
\rho_E : [0, \infty) \rightarrow [0, \infty),
$$

defined by

$$
\rho_E(t) = \sup\left\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : \|x\| \leq 1, \quad \|y\| \leq t\right\}.
$$

$E$ is said to be uniformly smooth if:

$$
\lim_{t \to 0} \frac{\rho_E(t)}{t} = 0.
$$

A uniformly smooth Banach space $E$ is said to have modulus of smoothness of power type $q > 1$ if there exists a constant $c > 0$ such that:

$$
\rho_E(t) \leq ct^q.
$$

It is well-known that all Hilbert spaces, and the Banach spaces $L_p$ (or $\ell_p$), $W^{p}_{m}$ and $e^p$, $1 < p < \infty$ have modulus of smoothness of certain power types (see e.g., [12], [25]). Hilbert spaces, $L_p$ (or $\ell_p$), $W^{p}_{m}$ and $e^p$ for $p \in [2, \infty)$ have modulus of smoothness of power type 2, while for $1 < p \leq 2$, $L_p$ (or $\ell_p$), $W^{p}_{m}$ and $e^p$ spaces have modulus of smoothness of power type $p$.

For $q > 1$ we shall denote by $J_q$ the generalized duality mapping from $E$ to $2^E^*$ given by

$$
J_q(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^q \text{ and } \|f^*\| = \|x\|^{q-1}\}.
$$
In particular $J = J_2$ is called the normalized duality map on $E$. It is well-known (see e.g., [25]) that $E$ is uniformly smooth if and only if $J$ (and hence $J_q$) is single valued and uniformly continuous on any bounded subset of $E$.

In his study of characteristic inequalities of uniformly smooth Banach spaces with modulus of smoothness of power type $q > 1$, Xu [23] proved the following theorem:

Theorem HKS ([23], Corollary 1’, P.1130): Let $E$ be a uniformly smooth Banach space. Then $E$ has modulus of smoothness of power type $q > 1$ if and only if there exists a constant $c > 0$ such that:

$$(7) \quad \|x + y\|^q \leq \|x\|^q + q \langle y, J_q(x) \rangle + c\|y\|^q$$

for all $x, y$ in $E$.

For Hilbert spaces $q = 2$, $c = 1$ and equality holds. For $p \geq 2$, $L_p$ (or $\ell_p$) spaces have modulus of smoothness of power type $q = 2$ and (7) is satisfied with $c = (p - 1)$ (see e.g., [23]). Also $L_p$ (or $\ell_p$) spaces, $p \in (1, 2]$ have modulus of smoothness of power type $q = p$ and hence satisfy (7). For an estimate of the constant $c$ for $L_p$ (or $\ell_p$), spaces $p \in (1, 2]$ the reader may consult [23].

In the sequel we shall need the following result:

**Lemma LQ ([15], p.302).** Let $\{x_n\}_{n=1}^{\infty}$ satisfy the following inequality:

$$x_{n+1} \leq \omega x_n + \sigma_n, \quad n \geq 1$$

where $x_n \geq 0, \sigma_n \geq 0$ and $\lim_{n \to \infty} \sigma_n = 0, 0 \leq \omega < 1$. Then $x_n \to 0$ as $n \to \infty$.

3. Main Results

We prove the following results:

**Lemma.** Let $E$ be a uniformly smooth Banach space with modulus of smoothness of power type $q > 1$. Then for all $x, y, z$ in $E$ and $\lambda \in [0, 1]$,

$$(8) \quad \|\lambda x + (1 - \lambda) y - z\|^q \leq [1 - \lambda(q - 1)] \|y - z\|^q + \lambda c\|x - z\|^q$$

$$- \lambda [1 - \lambda^{q-1} c]\|x - y\|^q$$

where $c > 0$ is the constant appearing in (7).
Proof. Using inequality (7) we obtain for all \( x, y, z \) in \( E \) and \( \lambda \in [0, 1] \);

\[
\|\lambda x + (1 - \lambda)y - z\|^q = \|y - z + \lambda(x - y)\|^q \\
\leq \|y - z\|^q + q\lambda\langle x - y, J_q(y - z) \rangle + c\lambda^q\|x - y\|^q.
\]

Furthermore,

\[
q\lambda\langle x - y, J_q(y - z) \rangle = q\lambda\langle x - z, J_q(y - z) \rangle - q\lambda\|y - z\|^q.
\]

and

\[
\|y - x\|^q = \|y - z + z - x\|^q \leq \|y - z\|^q + q\langle z - x, J_q(y - z) \rangle + c\|x - z\|^q
\]
i.e.,

\[
\|y - x\|^q \leq \|y - z\|^q - q\langle x - z, J_q(y - z) \rangle + c\|y - x\|^q
\]
so that,

\[
q\lambda\langle x - z, J_q(y - z) \rangle \leq \lambda\|y - z\|^q + \lambda c\|x - z\|^q - \lambda\|y - x\|^q.
\]

Substitution of (11) in (10) yields,

\[
q\lambda\langle x - y, J_q(y - z) \rangle \\
\leq \lambda\|y - z\|^q + \lambda c\|x - z\|^q - \lambda\|y - x\|^q
\]

\[
= -\lambda(q - 1)\|y - z\|^q + \lambda c\|x - z\|^q - \lambda\|y - x\|^q.
\]

Substituion of inequality (12) in (9) yields the desired result.

Remark 1. As has been observed, \( L_p \) (or \( \ell_p \)), spaces \( p \in [2, \infty) \) have modulus of smoothness of power type \( q = 2 \) and satisfy (7) for \( c = p - 1 \). By setting \( q = 2 \) and \( c = (p - 1) \) in (8) we obtain Lemma (9) of [5], for \( L_p \) spaces, \( p \in [2, \infty) \) (see also inequality (4) of [4]).

For the remainder of this paper, \( k \) will denote the constant appearing in definition of quasi-contractive map, and \( c \) will denote the constant appearing in our lemma. We prove the following theorems:
THEOREM 1. Let $E$ be a real uniformly smooth Banach space with modulus of smoothness of power type $q > 1$. Let $K$ be a closed, convex and bounded subset of $E$, and $T : K \to K$ a quasi-contractive mapping of $K$ into itself with $ck^q < \min(q - 1, 1)$. Let $\{c_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ be real sequences satisfying:

(i) $0 \leq \beta_n < 1, \quad n \geq 0$
(ii) $\lim_{n \to \infty} \beta_n = 0$.
(iii) $\frac{1}{2} \left( \frac{1}{c} (1 - ck^q) \right)^{\frac{1}{q}} \leq \alpha_n \leq \left( \frac{1}{c} (1 - ck^q) \right)^{\frac{1}{q}}$.

Then the sequence $\{x_n\}_{n=0}^\infty$ defined iteratively by,

$$x_0 = K,$$
$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n$$
$$y_n = (1 - \beta_n)x_n + \beta_n Tx_n, \quad n \geq 0$$

converges strongly to the unique fixed point of $T$.

Proof. From Ciric [8] and the conclusion of Rhoades [19], $T$ has a unique fixed point $x^*$ (say) in $K$. Set $y = x^*$ in (1) to obtain;

$$\|Tx - x^*\| \leq k \max \{\|x - x^*\|, \|x - Tx\|\}$$
for each $x \in K$. In particular,

$$\|Ty_n - x^*\| \leq k \max \{\|y_n - x^*\|, \|y_n - Ty_n\|\},$$
so that for all non-negative integers $n$, the following inequalities hold:

(13) $\|Ty_n - x^*\|^q \leq k^q (\|y_n - x^*\|^q + \|y_n - Ty_n\|^q)$
(14) $\|Tx_n - x^*\|^q \leq k^q (\|x_n - x^*\|^q + \|x_n - Tx_n\|^q)$.

Using inequality (8) and condition (ii), it follows that for $n$ sufficiently large,

$$\|Ty_n - x^*\|^q = \|(1 - \beta_n)(x_n - x^*) + \beta_n(Tx_n - x^*)\|^q$$
$$\leq [1 - \beta_n(q - 1)] \|x_n - x^*\|^q + c\beta_n\|Tx_n - x^*\|^q$$
$$- \beta_n(1 - c\beta_n^{q-1}) \|x_n - Tx_n\|^q$$
$$\leq \|x_n - x^*\|^q + c\beta_n\|Tx_n - x^*\|^q$$
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i.e.,

(15) \[ ||y_n - x^*||^q \leq ||x_n - x^*||^q + c\beta_n||Tx_n - x^*||^q \]

and,

\[
||y_n - Ty_n||^q = (1 - \beta_n)(x_n - Ty_n) + \beta_n(Tx_n - Ty_n)\]
\[
\leq [1 - \beta_n(q - 1)] ||x_n - Ty_n||^q + c\beta_n||Tx_n - Ty_n||^q
\]
\[
- \beta_n(1 - c\beta_n^{q-1})||x_n - Tx_n||^q
\]
\[
\leq [1 - \beta_n(q - 1)] ||x_n - Ty_n||^q + c\beta_n||Tx_n - Ty_n||^q
\]

i.e.,

(16) \[ ||y_n - Ty_n||^q \leq [1 - \beta_n(q - 1)] ||x_n - Ty_n||^q + c\beta_n||Tx_n - Ty_n||^q. \]

Substituting (15) and (16) in (13) yields;

\[
||Ty_n - x^*||^q \leq k^q||x_n - x^*||^q + c\beta_n||Tx_n - x^*||^q
\]
\[
+ k^q [1 - \beta_n(q - 1)] ||x_n - Ty_n||^q
\]
\[
+ k^q c\beta_n||Tx_n - Ty_n||^q.
\]

Setting \( D = 2ck^q(Diam K)^q \), this reduces, for sufficiently large \( n \), to:

(17) \[ ||Ty_n - x^*||^q \leq k^q||x_n - x^*||^q + k^q [1 - \beta_n(q - 1)] ||x_n - Ty_n||^q + \beta_n D \]

We can now estimate ||x_{n+1} - x^*||^q using (17)

\[
||x_{n+1} - x^*||^q = (1 - \alpha_n)(x_n - x^*) + \alpha_n(Ty_n - x^*)\]
\[
\leq [1 - \alpha_n(q - 1)] ||x_n - x^*||^q + c\alpha_n||Ty_n - x^*||^q
\]
\[
- \alpha_n[1 - c\alpha_n^{q-1}] ||x_n - Ty_n||^q
\]
\[
\leq [1 - \alpha_n(q - 1 - ck^q)] ||x_n - x^*||^q
\]
\[
- \alpha_n[1 - c\alpha_n^{q-1} - ck^q (1 - \beta_n(q - 1))] ||x_n - Ty_n||^q
\]
\[
+ \alpha_n\beta_n c D
\]
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i.e.,

\[
\|x_{n+1} - x^*\|^q \leq [1 - \alpha_n(q - 1 - ck^q)] \|x_n - x^*\|^q \\
- \alpha_n[1 - c\alpha_n^{q-1} - ck^q(1 - \beta_n(q - 1))]\|x_n - Ty_n\|^q \\
+ \alpha_n\beta_n M,
\]

where \( M = cD \).

Condition (iii) implies;

\[
1 - \alpha_n(q - 1 - ck^q) \leq 1 - \frac{1}{2}\frac{1}{c}(1 - ck^q) = h_q \text{ (say)} < 1,
\]

and,

\[
1 - c\alpha_n^{q-1} - ck^q(1 - \beta_n(q - 1)) \geq 1 - c\alpha_n^{q-1} - ck^q \geq 0,
\]

so that for sufficiently large \( n \), inequality (18) reduces to;

\[
\|x_{n+1} - x^*\|^q \leq h_q\|x_n - x^*\|^q + \alpha_n\beta_n M, \quad h_q \in (0, 1).
\]

Clearly, \( \alpha_n\beta_n M \to \infty \) and by setting \( \rho_n = \|x_n - x^*\|^q \), \( \omega = h_q \), and \( \sigma_n = \alpha_n\beta_n M \), it follows from lemma LQ that \( x_n \to x^* \) as \( n \to \infty \), completing the proof of Theorem 1.

**Remark 2.** For Hilbert spaces, \( q = 2 \) and \( c = 1 \), so that if we set \( q = 2 \), \( c = 1 \) in Theorem 1, then the condition \( ck^q < \min\{q - 1, 1\} \) reduces to \( k^2 < 1 \). Moreover, conditions (i), (ii) and (iii) reduce to exactly the same conditions of the Theorem of Qihou [15]. Thus, Theorem 1 extends the Theorem of [15] from Hilbert spaces to the more general Banach spaces considered here.

**Theorem 2.** Let \( E \) be a real uniformly smooth Banach space with modulus of smoothness of power type \( q > 1 \). Let \( K \) be a closed, convex and bounded subset of \( E \), and \( T : K \to K \) a quasi-contractive mapping of \( K \) into itself such that \( ck^q < \min\{q - 1, 1\} \). Let \( \{c_n\}_{n=0}^\infty \) be a real sequence satisfying

(i) \( 0 < c_n < 1, \quad n \geq 0 \)

(ii) \( \lim_{n \to \infty} c_n = 0 \)

(iii) \( \sum_{n=1}^\infty c_n = \infty \).
Then the sequence \( \{x_n\}_{n=0}^{\infty} \) defined iteratively by
\[
x_0 \in K \\
x_{n+1} = (1 - c_n)x_n + c_nTx_n, \quad n \geq 0,
\]
converges strongly to the unique fixed point of \( \Gamma \) in \( K \).

**Proof.** The existence of a unique fixed point follows as in Theorem 1. Again let \( x^* \) denote the fixed point. Then,
\[
\|x_{n+1} - x^*\|^q = \|(1 - c_n)(x_n - x^*) + c_n(Tx_n - x^*)\|^q \\
\leq [1 - c_n(q - 1)] \|x_n - x^*\|^q \\
+ cc_n \|Tx_n - x^*\|^q - c_n(1 - cc_n^{q-1}) \|x_n - Tx_n\|^q.
\]
Using inequality (14) this reduces to,
\[
\|x_{n+1} - x^*\|^q \leq [1 - c_n(q - 1 - ck^q)] \|x_n - x^*\|^q \\
- c_n(1 - cc_n^{q-1} - ck^q) \|x_n - Tx_n\|^q.
\]
Condition (ii) implies for some \( N_0 \) sufficiently large,
\[
N_0 \in \mathbb{N}, \quad 1 - c_n(q - 1 - ck^q) < 1 \text{ and } 1 - cc_n^{q-1} - ck^q \geq 0 \text{ so that}
\]
(19) \[
\|x_{n+1} - x^*\|^q \leq [1 - c_n(q - 1 - ck^q)] \|x_n - x^*\|^q.
\]
Iteration of inequality (19) from \( n = N_0 \) to \( N \) yields:
\[
\|x_{N+1} - x^*\|^q \leq \prod_{j=N_0}^{N} [1 - c_j(q - 1 - ck^q)] \|x_{N_0} - x^*\|^q \to 0
\]
as \( N \to \infty \), by condition (iii). Hence \( x_n \to x^* \) as \( n \to \infty \), completing the proof of Theorem 2.

**Remark 3.** Theorems 1 and 2 show that either the Mann iteration method or the Ishikawa iteration method can be used to approximate the fixed point of a quasi-contractive map in real uniformly smooth banach spaces with modulus of smoothness of power 1. However, the Mann iteration method may be preferred due to its simplicity.

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