MODULES WITH PRIME ENDOMORPHISM RINGS

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Abstract. Some discrimination of modules whose endomorphism rings are prime is introduced, by means of structures of submodules inducing prime ideals of the endomorphism ring \( \text{End}_R(M) \) of a left \( R \)-module \( RM \) over a ring \( R \). Modules with non-prime endomorphism rings are contrapositively studied as well.

1. Introduction

For any associative ring \( R \) and any left \( R \)-module \( RM \), its endomorphism ring \( \text{End}_R(M) \) will act on the right side of \( RM \), in other words, \( RM \text{End}_R(M) \) will be studied mainly. Thus the composite of functions preserves the order such that the composite

\[ fg : A \xrightarrow{f} B \xrightarrow{g} C \]

of \( f : A \to B \) and \( g : B \to C \) defined by \( afg = (af)g \) for every \( a \in A \). Without conflict, for any mapping \( f : M \to N \), \( K \subseteq M, L \subseteq N \) we also frequently will use notations of the image \( f(K) = Kf \) of \( K \) under \( f \) and the preimage \( f^{-1}(L) = Lf^{-1} \) of \( L \) under \( f \) as usual.

For any left \( R \)-module \( RM \), the endomorphism ring \( \text{End}_R(M) \) is said to be a prime ring if \( fg = 0 \) implies that \( f = 0 \) or \( g = 0 \). If \( fg = 0 \) with an epimorphism \( f \) or a monomorphism \( g \), then \( f = 0 \) or \( g = 0 \) follows. For instance, if every nonzero endomorphism \( f : RM \to RM \) is a monomorphism(or an epimorphism), then it clearly follows that \( \text{End}_R(M) \) is a prime ring. However there are some modules satisfying none of these. In order to study these modules having prime endomorphism rings we need some definitions of submodules of modules.

For any subset \( J \) of \( \text{End}_R(M) \), let \( \text{Im}J = MJ = \sum_{f \in J} \text{Im}f \) and \( \text{ker}J = \cap_{f \in J} \text{ker}f \) be the sum of images of endomorphisms in \( J \) and the intersection of kernels of endomorphisms in \( J \), respectively. Also we call \( N \) an open submodule if \( N = N^o \), where \( N^o = \sum_{f \in S} \text{Im}f \leq N \text{Im}f \) is the sum of all images...
of endomorphisms contained in $N$ and call $N$ a closed submodule if $N = N$, where $N = \bigcap_{f \in S, N \subseteq \ker f} \ker f$ is the intersection of all kernels of endomorphisms containing $N$, and where $S = \text{End}_R(M)$.

Here are some simple and easy conditions for any module $R^M$ to have a prime endomorphism ring:

1. If each nonzero open submodule $A$ is isomorphic or equal to $M$, it clearly follows that the endomorphism ring $\text{End}_R(M)$ is a prime ring.
2. If each nonzero closed submodule is isomorphic or equal to $M$, then the endomorphism ring $\text{End}_R(M)$ is a prime ring.

However these kinds of definitions would give non-enough informations of prime endomorphism rings. Here are other definitions of submodules inducing prime ideals of endomorphism rings which was studied in [6]. Some results from [6] are written in this section.

**Definition 1.1 ([6]).** For a submodule $P \leq M$ of a left $R$-module $R^M$, $P$ is said to be a meet-prime submodule of $R^M$ if it satisfies the following conditions; for any open submodules $A, B \leq M$ with $P^o + A \neq M$ or $P^o + B \neq M$,

1. if $A \cap B \leq P$, then $A \leq P$ or $B \leq P$,
2. if $(P \cap A \cap B)^o \neq 0$, then $A \leq P$ or $B \leq P$,
3. if $P \cap A = 0$, then $A = 0$ or $P + A = M$.

A module $R^M$ is said to be meet-prime if the trivial submodule $0$ of $R^M$ is meet-prime.

In particular, if the trivial submodule $0 \leq M$ of a module $R^M$ satisfies the item (1), then we will call the trivial $0$ a quasi-meet-prime submodule(or meet-irreducible in terms of open submodules)of $R^M$, or will call $R^M$ a quasi-meet-prime module.

**Definition 1.2.** For a left $R$-module $R^M$, $0 \leq M$ is said to be a $\cap$-prime(or intersection-prime, or cap-prime) submodule of $R^M$ if it satisfies the following conditions: for any open submodules $A, B \leq M$,

1. if $A \cap B \leq 0$, then $A = 0$ or $B = 0$,
2. $A = 0$, or $A$ is isomorphic or equal to $M$ (briefly, denoted by $A \simeq M$).

A module $R^M$ is said to be $\cap$-prime if the trivial submodule $0$ of $R^M$ is $\cap$-prime.

Clearly in any module if $0$ is meet-prime, then $0$ is $\cap$-prime, in other words, every meet-prime module is a $\cap$-prime module. However the converse
is not true in general, for example, the integer ring \( \mathbb{Z} \) has the trivial \( 0 \leq \mathbb{Z} \) is a \( \cap\)-prime submodule \( 0 \leq \mathbb{Z} \) but not a meet-prime submodule of it.

Easily for any submodule \( P \leq M \), we have that \( P \) is meet-prime if and only if \( P^{\cap} \) is meet-prime and that every module isomorphism preserves the meet-primeness and the \( \cap\)-primeness between isomorphic modules.

Recall a module \( R \) is said to be simple if all submodules of \( R \) are only the trivial submodules \( 0 \) and \( M \) itself. Likewise, we define a module \( R \) to be openly simple by all open submodules of \( R \) are only the trivial submodules \( 0 \) and \( M \) itself.

**Remark 1.3.** Any simple module is openly simple, however the converse is not true in general. For the integer ring \( \mathbb{Z} \), a left \( \mathbb{Z} \)-module \( \mathbb{Z}(p^{\infty}) \) for prime \( p \) is openly simple but not simple.

**Lemma 1.4.** For any left \( R \)-module \( R \), we have that \( 0 \leq M \) is meet-prime in \( R \) if and only if \( R \) is openly simple.

Hereafter \( S \) denotes the endomorphism ring \( \text{End}_R(M) \) of a left \( R \)-module \( R \).

**Lemma 1.5.** For any left \( R \)-module \( R \), we have the following:

1. If \( P \leq M \) is any fully invariant meet-prime submodule of \( R \), then \( I^P = \{ f \in S \mid \text{Im} f \leq P \} \leq S \) is a prime ideal of \( S \).
2. If \( 0 \leq M \) is a \( \cap\)-prime submodule of \( R \), then \( 0 \leq S \) is a prime ideal of \( S \), that is, \( S \) is a prime ring.

**Proposition 1.6.** For any left \( R \)-module \( R \), if at least one of the following is satisfied:

1. \( R \) is an openly simple module.
2. For each nonzero endomorphism \( f : R \rightarrow R \), \( (\ker f)^o = 0 \).
3. Every nonzero open submodule is isomorphic or equal to \( M \).
4. Every open submodule of \( R \) is fully invariant essential(or large) and \( 0 \leq M \) is quasi-meet-prime.
5. \( S \) is commutative and \( 0 \leq M \) is quasi-meet-prime.
6. The zero submodule \( 0 \leq M \) is \( \cap\)-prime.

Then the endomorphism ring \( S \) is a prime ring.

A left \( R \)-module \( R \) is said to be self-generated if each submodule of \( R \) is open ([4]). It is clear that for any self generated module \( R \), \( 0 \) is meet-prime if and only if \( R \) is simple.
**Definition 1.7** ([6]). For a submodule $P \leq M$ of a left $R$–module $R^M$, we will say that $P$ is a sum-prime submodule of $R^M$ if it satisfies the following conditions: for any *closed* submodules $A, B \leq M$ with $P \cap A \neq 0$ or $P \cap B \neq 0$,

1. if $P \leq A + B$, then $P \leq A$ or $P \leq B$,
2. if $P + A + B \neq M$, then $P \leq A$ or $P \leq B$,
3. if $P + A = M$, then $A = M$ or $P \cap A = 0$.

A module $R^M$ is said to be *sum-prime* if $M$ is a sum-prime submodule of $R^M$. In particular, if the trivial submodule $M$ of a module $R^M$ satisfies the item (1), then we will call $R^M$ *quasi-sum-prime* (or sum-irreducible in terms of closed submodules).

**Definition 1.8.** For a left $R$–module $R^M$, we will say that $M$ is a $+$-prime submodule of $R^M$ if it satisfies the following conditions: for any *closed* submodules $A, B \leq M$,

1. if $M \leq A + B$, then $M = A$ or $M = B$,
2. $A = 0$ or $A \cong M$ is isomorphic or equal to $M$.

A module $R^M$ is said to be $+$-prime if $M$ is a $+$-prime submodule of $R^M$.

Clearly for any submodule $P \leq M$, we have that $P$ is a sum-prime submodule of $R^M$ if and only if $P$ is a sum-prime submodule of $R^M$ and that every module isomorphism preserves the sum-primeness and the $+$-primeness between isomorphic modules. We also have that every sum-prime module is a $+$-prime module.

We also define a module $R^M$ to be *closedly simple* by all the closed submodules of $R^M$ are the trivial submodules $0$ and $M$ only.

**Remark 1.9.** Any simple module is also closedly simple, however the converse is not true in general. For the integer ring $\mathbb{Z}$, a left $\mathbb{Z}$–module $\mathbb{Z}\mathbb{Z}$ is closedly simple but not simple.

**Lemma 1.10.** For any left $R$–module $R^M$, we have that $M$ is sum-prime in $R^M$ if and only if $R^M$ is closedly simple.

**Lemma 1.11** ([6]). For any left $R$–module $R^M$, we have the following.

1. If $P \leq M$ is any fully invariant sum-prime submodule of $R^M$, then $I_P = \{ f \in S \mid P \leq \ker f \}$ is a prime ideal of $S$.
2. If $M$ is a $+$-prime submodule of $R^M$, then $0 \not\leq S$ is a prime ideal of $S$, that is, $S$ is a prime ring.
Proposition 1.12. For any left $\mathcal{R}$−module $\mathcal{R} M$, if at least one of the following is satisfied:

1. $\mathcal{R} M$ is a closedly simple module.
2. For each nonzero endomorphism $f : \mathcal{R} M \rightarrow \mathcal{R} M$, $\operatorname{Im} f = \mathcal{M} f = M$ is improper, i.e., $\operatorname{Im} f = \mathcal{M} f = M$.
3. Every nonzero closed submodule is isomorphic or equal to $\mathcal{M}$.
4. Every closed submodule of $\mathcal{R} M$ is fully invariant superfluous(or small) and $\mathcal{M} \leq \mathcal{R} M$ is quasi-sum-prime.
5. $S$ is commutative and $\mathcal{M} \leq \mathcal{R} M$ is quasi-sum-prime.
6. The trivial submodule $\mathcal{M} \leq \mathcal{R} M$ is $+^\prime$-prime.

Then the endomorphism ring $S$ is a prime ring.

A left $\mathcal{R} M$ is said to be self-cogenerated if each submodule of $\mathcal{R} M$ is closed ([4]). It is clear that any self cogenerated sum-prime module is simple.

2. Meet-prime or $\cap$-prime submodules under homomorphisms

For any function $f : \mathcal{R} M \rightarrow \mathcal{R} N$ the preimage assignment of $f$, conveniently denoted by $f^{-1}$ or $f^\leftarrow : \mathcal{P}(N) \rightarrow \mathcal{P}(M)$ from the power set $\mathcal{P}(N)$ of $\mathcal{R} N$ into the power set $\mathcal{P}(M)$ of $\mathcal{R} M$ is a function always.

An $\mathcal{R}$−homomorphism $f : \mathcal{R} M \rightarrow \mathcal{R} N$ is said to be open if the image assignment $f : \mathcal{P}(M) \rightarrow \mathcal{P}(N)$ preserves open submodules, in other words, $f(A) \leq N$ is an open submodule of $\mathcal{R} N$, for any open submodule $A \leq M$.

Theorem 2.1. For any open monomorphism $f : \mathcal{R} M \rightarrow \mathcal{R} N$, we have the following:

1. If $P$ is a meet-prime submodule of $\mathcal{R} N$, then $f^{-1}(P)$ is also a meet-prime submodule of $\mathcal{R} M$.
2. If $\mathcal{R} N$ is a $\cap$-prime module, then $\mathcal{R} M$ is also a $\cap$-prime module.

Proof. (1) For any open submodules $A, B \leq M$ such that $f^{-1}(P) + A \neq M$ or $f^{-1}(P) + B \neq M$, (i) if $A \cap B \leq f^{-1}(P)$, then since $f$ is a monomorphism $f(A \cap B) = f(A) \cap f(B) \leq P$. Since $f$ is open and $P$ is meet(resp. $\cap$)-prime in $\mathcal{R} N$, it follows that $f(A) \leq P$ or $f(B) \leq P$. Therefore $A \leq f^{-1}(P)$ or $B \leq f^{-1}(P)$.

(ii) If $[A \cap B \cap f^{-1}(P)]^\circ \neq 0$, then $A \cap B \cap f^{-1}(P) \neq 0$ follows immediately. From the openness of the monomorphism $f$ it follows easily that $0 \neq f(A) \cap f(B) \cap f(f^{-1}(P)) \leq f(A) \cap f(B) \cap P$, $f(A), f(B)$ are open submodules of...
such that $P^o + f(A) \neq N$ or $P^o + f(B) \neq N$. From the meet(resp. $\cap$)-primeness of $P$ it follows that $f(A) \leq P$ or $f(B) \leq P$ and hence $A \leq f^{-1}(P)$ or $B \leq f^{-1}(P)$.

(iii) If $A \cap f^{-1}(P) = 0$ (resp. with $f^{-1}(P) \neq 0$), then from a monomorphism $f$ it follows that $f(A) \cap P = 0$.

Thus $f(A) = 0$ or $P + f(A) = N$ follows from the meet (resp. $\cap$)-primeness of $P$. Hence we have clearly that $A = 0$ or $f^{-1}(P) + A = M$.

(2): If $f^{-1}(P) = 0$, then $P \cap f(M) = 0$. For the case of $P \neq 0$ we have that $P + f(A) = P \oplus f(A) = N$ and hence $A = f^{-1}(N) = M$. For the case of $P = 0$ we have that $f(A) = 0$ or $f(A) \simeq N$. Since $f(M) \leq N$ is an open submodule of $N$ we have that $f(M) = 0$ or $f(M) \simeq N$. Therefore $f(A) = 0$ or $f(A) \simeq f(M)$ and hence $A = 0$ or $A \simeq M$.

\begin{corollary}
For any monomorphism $f : R \to N$ with a self-generated module $N$, we have the following.

1. If $P$ is a meet-prime submodule of $N$, then $f^{-1}(P)$ is also a meet-prime submodule of $N$.
2. If $N$ is a $\cap$-prime module, then $N$ is also a $\cap$-prime module.

\end{corollary}

Proof. Since for any self-generated module $N$ any homomorphism $f : R \to N$ is an open mapping. Thus the proof is completed by the same proof of Theorem 2.1. \qed

\begin{remark}
It is careful to apply the above Theorem 2.1 to the inclusion mapping $\iota : R K \subseteq N$. Since for any submodule $K \leq M$, the open submodule $A = \sum_{g \in \operatorname{End}_R(K), K g \leq A} K g \neq \sum_{f \in \operatorname{End}_M(M), M f \leq A} M f$, in general. In other words, it is not necessary for all open submodules in any submodule $R K \leq R M$ to be open submodules of $R M$. For example, for any prime number $p$, a module $\mathbb{Z}(p^\infty)$ having a submodule $K = \{ 0, 1/p, 2/p, \ldots, (p - 1)/p, 1/p^2, 2/p^2, \ldots, (p - 1)/p^2 \} \leq \mathbb{Z}(p^\infty)$ is such a module that the inclusion mapping $\iota : \mathbb{Z}(p^\infty) \subseteq \mathbb{Z}(p^\infty)$ is not an open mapping.

\end{remark}

\begin{corollary}
For any module $R M$ and for a submodule $K \leq M$ such that each open submodule $A \leq K$ of $R K$ is open in $R M$, that is, $A = \sum_{g \in \operatorname{End}_R(K), K g \leq A} K g = \sum_{f \in \operatorname{End}_M(M), M f \leq A} M f$, we have the following.

1. If $P$ is a meet-prime submodule of $R M$, then $P \cap K$ is meet-prime in $R K$.
2. If $N$ is a $\cap$-prime module, then $N$ is also a $\cap$-prime module.

\end{corollary}
Proof. Since the inclusion $\iota: R K \to R M$ is an open monomorphism by Theorem 2.1, we have that $P \cap K$ is a meet (resp. $0 \leq K$ is $\cap$)-prime submodule of $R K$. □

**Corollary 2.5.** For any $\cap$-prime module $R M$ and for a submodule $K \leq M$ such that each open submodule

$$A = \sum_{g \in \text{End}_R(K), Kg \leq A} Kg = \sum_{f \in \text{End}_R(M), Mf \leq A} Mf,$$

we have a $\cap$-prime module $R K$ and furthermore $\text{End}_R(K)$ is a prime endomorphism ring.

Proof. Considering the inclusion mapping $\iota: R K \hookrightarrow R M$, then we have a monomorphism $\iota$ such that $\ker \iota = 0$ is also $\cap$-prime in $R K$ by the $\cap$-primeness of 0 in $R M$. Hence the endomorphism ring $\text{End}_R(K)$ is prime. □

**Corollary 2.6.** For a self-generated $\cap$-prime module $R M$ and for any submodule $K \leq M$, we have a $\cap$-prime module $R K$ and furthermore $\text{End}_R(K)$ is a prime endomorphism ring.

Proof. Since the inclusion mapping $\iota: R K \hookrightarrow R M$ with a self-generated module $R M$ is an open monomorphism always. From Corollary 2.5 it follows that $R K$ is also a $\cap$-prime module, i.e., $0 \leq K$ is $\cap$-prime and hence the endomorphism ring $\text{End}_R(K)$ is a prime ring. □

**Theorem 2.7.** For any $R$-epimorphism $f: R M \to R N$ with the open preimage assignment and for a submodule $Q \leq N$ of $R N$, we have the following.

1. If $f^{-1}(Q) \leq M$ is meet-prime, then $Q$ is a meet-prime submodule of $R N$.
2. If $\ker f \leq M$ is a meet-prime submodule of $R M$, then $R N$ is a meet-prime module, and furthermore we have a meet-prime quotient module $R M/\ker f$. 
Corollary 2.8. For any $R$-epimorphism $f: RM \to RN$ with a self generated module $RM$ and for a submodule $Q \leq N$ of $RN$, we have the following.

(1) If $f^{-1}(Q) \leq M$ is meet-prime, then $Q$ is a meet-prime submodule of $RN$.

(2) If $\ker f \leq M$ is a meet-prime submodule of $RM$, then $RN$ is a meet-prime module. Furthermore we have a meet-prime quotient module $RM/\ker f$.

Proof. Since $RM$ is self-generated module any homomorphism $RM \to RN$ has the open preimage assignment. By Theorem 2.7 the proof is established easily. \[\square\]

For any module $RM$ and for any submodule $K \leq M$ of $RM$, considering the quotient module $RM/K$ and the projection $\pi: RM \to RM/K$, additionally if $K$ is open and fully invariant, then the projection $\pi: RM \to RM/K$ has an open image assignment, i.e. we have an open submodule $\pi(A) \leq M/K$ for any open submodule $A \leq M$ such that $A \supseteq K$.

Remark 2.9. However the projection $\pi$ doesn’t have an open preimage assignment in general. For example, let $\mathbb{Z}Q$ be the $\mathbb{Z}$-module of rational numbers over the integer ring $\mathbb{Z}$. Then $\pi: \mathbb{Z}Q \to \mathbb{Z}Q/\mathbb{Z}$ doesn’t have an open preimage assignment.

We have an immediate consequence of the above Theorem 2.7 that the meet-primeness is cohereditary in a kind of the factor modules.

Corollary 2.10. For any module $RM$ and for any fully invariant open submodule $K \leq M$ of $RM$, if $P \leq M$ such that $K \leq P$ and if $\pi(P) \leq M/K$ is meet-prime, then $P$ is a meet-prime submodule of $RM$.

Corollary 2.11. For any module $RM$ and for any open fully invariant submodule $K$ of $RM$, if the quotient module $RM/K$ is meet-prime, then $K \leq M$ is meet-prime.

Proof. Since the projection mapping $\pi: RM \to RM/K$ has an open image assignment for each open fully invariant submodule $K \leq M$. Additionally if $0 = K \leq M/K$ is meet-prime in $RM/K$, then we have immediately that $K \leq M$ is a meet-prime submodule of $RM$. \[\square\]
Theorem 2.12. For a self-generated module $R\mathcal{N}$ and for any $R$-epimorphism $f : R\mathcal{M} \rightarrow R\mathcal{N}$, if $P \leq \mathcal{N}$ is meet-prime in $R\mathcal{N}$, then $f^{-1}(P) \leq \mathcal{M}$ is a meet-prime submodule of $R\mathcal{M}$.

Proof. For any $R$–homomorphism $f : R\mathcal{M} \rightarrow R\mathcal{N}$ with a self-generated module $R\mathcal{N}$, we have the induced isomorphism $\overline{f} : R\mathcal{M}/\ker f \rightarrow R\mathcal{N}$ of $f : R\mathcal{M} \rightarrow R\mathcal{N}$. From the self-generatedness of $R\mathcal{N}$ it follows that $R\mathcal{M}/\ker f$ is also a self-generated module. Thus the projection $\pi : R\mathcal{M} \rightarrow R\mathcal{M}/\ker f$ is an open epimorphism.

Now that $P \leq \mathcal{N}$ is meet-prime if and only if $f^{-1}(P) \leq \mathcal{M}/\ker f$ is meet-prime it remains to show that $f^{-1}(P) \leq \mathcal{M}$ is meet-prime for any given meet-prime submodule $P \leq \mathcal{N}$.

By the Corollary 2.10 it immediately concludes that $f^{-1}(P) \leq \mathcal{M}$ is meet-prime if $P \leq \mathcal{N}$ is a meet-prime submodule of $R\mathcal{N}$. □

Corollary 2.13. For self-generated modules $R\mathcal{M}, R\mathcal{N}$, for any submodule $P \leq \mathcal{N}$ of $R\mathcal{N}$, and for any $R$–epimorphism $f : R\mathcal{M} \rightarrow R\mathcal{N}$, the following are equivalent:

1. $P \leq \mathcal{N}$ is a meet-prime submodule of $R\mathcal{N}$;
2. $f^{-1}(P) \leq \mathcal{M}$ is a meet-prime submodule of $R\mathcal{M}$.

Proof. By the Corollary 2.10 and by the Theorem 2.12 the proof is completed at once. □

3. Sum-prime or $+$prime submodules under homomorphisms

An $R$–homomorphism $f : R\mathcal{M} \rightarrow R\mathcal{N}$ is said to be closed if the image assignment $f : \mathcal{P}(M) \rightarrow \mathcal{P}(N)$ preserves closed submodules, in other words, $f(A) \leq \mathcal{N}$ is a closed submodule of $R\mathcal{N}$, for any closed submodule $A \leq \mathcal{M}$. For example, any inclusion mapping $\iota : n\mathbb{Z} \hookrightarrow \mathbb{Z}$ (for any $n \in \mathbb{N}$) is a closed monomorphism. We have some results for sum-prime submodules.

Theorem 3.1. For any closed monomorphism $f : R\mathcal{M} \rightarrow R\mathcal{N}$ and for a submodule $Q \leq \mathcal{M}$ of $R\mathcal{M}$, we have the following.

1. If $f(Q)$ is sum-prime in $R\mathcal{N}$, then $Q$ is sum-prime in $R\mathcal{M}$.
2. If $R\mathcal{N}$ is $+$ (or sum $-$) prime, then $R\mathcal{M}$ is $+$ (or sum $-$) prime, respectively.
Proof. (1): It is elementary.

(2): (ii) It first is going to show that for any closed submodule \( A \leq M \), \( A = 0 \) or \( A \cong M \). Since the improper submodule \( M = \ker 0 \) is a closed submodule of \( R M \) we have a closed submodule \( f(M) \leq N \). From the +primeness of \( N \leq N \) it follows that \( f(A) = 0 \) or \( f(A) \cong N \) for any closed submodule \( A \leq M \) and also we have \( f(M) \cong N \). Hence we have that \( A = 0 \) or \( A \cong M \) by the monomorphism \( f \). (i) if \( M \leq A + B \) with closed submodules \( A, B \) such that \( A \neq 0 \) or \( B \neq 0 \), then \( f(M) \leq f(A + B) = f(A) + f(B) \) with all closed submodules \( f(M), f(A) + f(B), f(A), f(B) \leq N \) and \( f(M) \cong N, f(A) + f(B) \cong N, f(A) \cong N \) or \( f(B) \cong N \). Since \( N \) is \( \cap \)-prime in \( R N \) we have that \( f(M) \leq f(A) \) or \( f(M) \leq f(B) \). Then it follows that \( M \leq A \) or \( M \leq B \). Therefore \( R M \) is a +prime module. For the case of a sum-prime module \( R N \), the similar method by replacing \( \cong \) by \( = \) completes the proof. \( \Box \)

**Corollary 3.2.** For any monomorphism \( f : R M \rightarrow R N \) with a self-cogenerated module \( R N \) and for a submodule \( Q \leq M \) of \( R M \), we have the following.

(1) If \( f(Q) \) is sum-prime in \( R N \), then \( Q \) is sum-prime in \( R M \).

(2) If \( R N \) is +\( ( \) or sum\( - \)\)prime, then \( R M \) is +\( ( \) or sum\( - \)\)prime, respectively.

**Proof.** Since any homomorphism \( f : R M \rightarrow R N \) with a self-cogenerated module \( R N \) is a closed mapping, especially for any closed submodule \( A \leq M \) we have that \( f(A) \leq N \) is a closed submodule of a self-cogenerated module \( R N \). Thus the proof is completed by Theorem 3.1. \( \Box \)

** Remark 3.3.** It is careful to apply the above Theorem 3.1 to the inclusion mapping \( \iota : R K \hookrightarrow R M \). Since any closed submodule \( A = \cap_{g \in \End_{\mathbb{K}}(K); A \leq \ker g} \ker f \) of a submodule \( \mu K ( \leq R M ) \) need not to be a closed submodule of \( \mu M \), in general. In other words, it is not necessary for all closed submodules in \( R K ( \text{for } K \leq M ) \) to be closed submodules of \( R M \). For example, a module \( \mathbb{Z}Q \) having a submodule \( \mathbb{Z} \mathbb{Z} \leq \mathbb{Z}Q \) is such a module that the inclusion mapping \( \iota : \mathbb{Z} \mathbb{Z} \hookrightarrow \mathbb{Z}Q \) is not a closed mapping.

**Corollary 3.4.** For any module \( R M \) and for a submodule \( K \leq M \), if the inclusion mapping \( \iota : R K \hookrightarrow R M \) is a closed monomorphism, then we have the following.

(1) If \( f(Q) \) is sum-prime in \( R M \), then \( Q \) is a sum-prime submodule of \( R K \).
(2) If \( R \mathcal{N} \) is \(+\)-(or sum-)prime, then \( R \mathcal{M} \) is \(+\)-(or sum-)prime, respectively.

Proof. It is an immediate consequence of Theorem 3.1. \( \square \)

**Corollary 3.5.** For any module \( R \mathcal{M} \) and for a submodule \( K \leq M \), if the inclusion mapping \( \iota : R \mathcal{K} \hookrightarrow R \mathcal{M} \) is a closed monomorphism, then we have the following.

1. If \( K \) is sum-prime in \( R \mathcal{M} \), then \( K \) is sum-prime in \( R \mathcal{K} \) and hence \( \text{End}_{R}(K) \) is prime.
2. If \( R \mathcal{M} \) is \(+\)-(or sum-)prime, then \( R \mathcal{K} \) is \(+\)-(or sum-)prime, respectively.

Proof. Since a submodule \( K \leq M \) is sum-prime if and only if \( \overline{K} \leq M \) is sum-prime and since the inclusion mapping \( \iota : R \mathcal{K} \to R \mathcal{M} \) is a closed monomorphism it follows quickly from Theorem 3.1 that \( K \) is sum-prime in \( R \mathcal{K} \). Furthermore we have a prime endomorphism ring \( \text{End}_{R}(K) \). \( \square \)

**Corollary 3.6.** For any self-cogenerated module \( R \mathcal{M} \) and any submodule \( K \leq M \), we have the following.

1. If \( K \) is sum-prime in \( R \mathcal{M} \), then \( K \) is sum-prime in \( R \mathcal{K} \) and hence the endomorphism ring \( \text{End}_{R}(K) \) is prime.
2. Additionally if \( R \mathcal{M} \) is \(+\)-(or sum-)prime, then every submodule \( R \mathcal{K} \) is \(+\)-(or sum-)prime, respectively. And hence we have a prime endomorphism ring \( \text{End}_{R}(K) \).

Proof. Since every submodule \( K \leq M \) is a closed submodule of \( R \mathcal{M} \) every closed submodule of \( R \mathcal{K} \) is also a closed submodule of a self-cogenerated module \( R \mathcal{M} \) and thus we have that the inclusion \( \iota : R \mathcal{K} \hookrightarrow R \mathcal{M} \) is a closed monomorphism. By Theorem 3.1 we have that \( K \) is sum-prime in \( R \mathcal{K} \). Therefore the prime endomorphism \( \text{End}_{R}(K) \) is obtained automatically. \( \square \)

**Theorem 3.7.** For any epimorphism \( f : R \mathcal{M} \to R \mathcal{N} \) with the closed preimage assignment and for a submodule \( P \leq N \) of \( R \mathcal{N} \), we have the following.

1. If \( f^{-1}(P) \) is a sum-prime submodule of \( R \mathcal{M} \), then \( P \) is also a sum-prime submodule of \( R \mathcal{N} \).
2. If \( R \mathcal{M} \) is a \(+\)-(or sum-)prime module, then \( R \mathcal{N} \) is \(+\)-(or sum-) prime, respectively.
Proof. (1): It is sufficient to show that $\mathcal{P}$ is a sum-prime submodule of $R N$. For any closed submodule $C \leq N$ with $\mathcal{P} \cap C \neq 0$, we have that $f^{-1}(\mathcal{P}) \cap f^{-1}(C) \neq 0$. And $f^{-1}(\mathcal{P}) \cap f^{-1}(C) = f^{-1}(\mathcal{P}) \cap f^{-1}(C) \neq 0$ follows from the closed preimage assignment of $f$.

For any closed submodules $A, B \leq N$ with $\mathcal{P} \cap A \neq 0$ or $\mathcal{P} \cap B \neq 0$, we also have that $f^{-1}(\mathcal{P}) \cap f^{-1}(A) \neq 0$ or $f^{-1}(\mathcal{P}) \cap f^{-1}(B) \neq 0$.

(i) If $P \leq A + B$, then $\mathcal{P} \leq A + B$ since $A + B = \ker(I_A \cap I_B)$ is a closed submodule of $R N$. Thus $f^{-1}(\mathcal{P}) \leq f^{-1}(A + B) = f^{-1}(A) + f^{-1}(B) = f^{-1}(A) + f^{-1}(B)$ implies that $f^{-1}(\mathcal{P}) \leq f^{-1}(A)$ or $f^{-1}(\mathcal{P}) \leq f^{-1}(B)$ by the sum-primefulness of $f^{-1}(\mathcal{P})$. Thus it follows from an epimorphism $f$ that $\mathcal{P} \leq A$ or $\mathcal{P} \leq B$.

(ii) If $\mathcal{P} + A + B \neq N$, then the closed submodule $f^{-1}(\mathcal{P}) + f^{-1}(A) + f^{-1}(B) \neq M$ follows. $f^{-1}(\mathcal{P}) + f^{-1}(A) + f^{-1}(B) \neq M$ by the sum-primefulness of $f^{-1}(\mathcal{P})$ implies that $f^{-1}(\mathcal{P}) \leq f^{-1}(A)$ or $f^{-1}(\mathcal{P}) \leq f^{-1}(B)$. Thus $\mathcal{P} \leq A$ or $\mathcal{P} \leq B$ follows immediately.

(iii) If $\mathcal{P} + A = N$, then $f^{-1}(\mathcal{P}) + f^{-1}(A) = f^{-1}(\mathcal{P}) + f^{-1}(A) = M$. By the sum-primefulness of $f^{-1}(\mathcal{P})$ it follows that $f^{-1}(A) = M$ or $f^{-1}(\mathcal{P}) \cap f^{-1}(A) = 0$. Thus $A = N$ or $\mathcal{P} \cap A = 0$ follows. Therefore $\mathcal{P}$ is sum-prime and hence $P$ is sum-prime in $R N$.

(2): For any nonzero closed submodules $A, B \leq N$, we have closed submodules $f^{-1}(A) \simeq M$ or $f^{-1}(B) \simeq M$ since $M$ is +prime. Hence it follows clearly that $A \simeq f(M) = N$ or $B \simeq f(M) = N$. The rest of the proof are completed by the same methods done in the proof of (2) of Theorem 3.1. □

Corollary 3.8. For any epimorphism $f : R M \rightarrow R N$ with a self-cogenerated module $R M$ and for a submodule $P \leq N$ of $R N$, we have the following.

(1) If $f^{-1}(P)$ is a sum-prime submodule of $R M$, then $P$ is also a sum-prime submodule of $R N$.

(2) If $R M$ is $(+ or \text{ sum-})$prime, then $R N$ is $(+ or \text{ sum-})$prime, respectively.

Proof. Since the preimage assignment of $f : R M \rightarrow R N$ for any self-cogenerated module $R M$ is closed by Theorem 3.8 the proof is completed. □

Remark 3.9. The preimage assignment $A + K \mapsto \pi^{-1}(A + K) = A$ of the projection $\pi : R M \rightarrow R M/K$ for each submodule $A \leq M$ is not necessary to be closed, in general.

However if $A \leq M$ is a closed submodule of $R M$, then it follows easily that $A + K$ is also a closed submodule of $R M$ which doesn’t guarantee that $A + K$
is a closed submodule of $R_{M/K}$ for any submodule $K \leq M$. For example, for the Abelian group $\mathbb{Q}$ of rational numbers, considering a module $\mathbb{Z}\mathbb{Q}$ (forget the multiplication in $\mathbb{Q}$) with a submodule $\mathbb{Z}$ and a quotient module $\mathbb{Z}\mathbb{Q}/\mathbb{Z}$. Then each submodule $\mathbb{Z}(p^\infty)$ (for any prime number $p$) of $\mathbb{Z}\mathbb{Q}/\mathbb{Z}$ is a closed submodule but the preimage of $\mathbb{Z}(p^\infty)$ under the projection $\pi: \mathbb{Z}\mathbb{Q} \to \mathbb{Z}\mathbb{Q}/\mathbb{Z}$ is not closed in $\mathbb{Z}\mathbb{Q}$.

**Corollary 3.10.** For any module $R_{M}$ and for any closed fully invariant submodule $K \leq M$ such that the projection $\pi: R_{M} \to R_{M/K}$ has a closed preimage assignment, we have the following.

1. If $G \leq M$ is a sum-prime submodule of $R_{M}$, then $\pi(G) \leq M/K$ is also sum-prime.
2. If $Q \leq M/K$ is a sum-prime submodule of $R_{M/K}$, then $\pi^{-1}(Q) \leq M$ is also a sum-prime submodule of $R_{M}$.

**Proof.** For any fully invariant submodule $K \leq M$ we have the closed image assignment of $\pi: R_{M} \to R_{M/K}$.

The item (1) follows directly from the closed image assignment of $\pi$ since $K \leq M$ is closed fully invariant.

(2): With the closed image assignment of $\pi: R_{M} \to R_{M/K}$ Theorem 3.7 tells that if $\pi^{-1}(F) \leq M$ is sum-prime in $R_{M}$, then $F \leq M/K$ is a sum-prime submodule of $R_{M/K}$. □

4. Modules with non-prime endomorphism rings

Properties of nonprime endomorphism rings of modules are studied by using images and kernels of endomorphisms in this section 4, and we will see some contraposition of statements for modules whose endomorphism rings are prime.

4.1. Using images of kernels of endomorphisms

For any endomorphism ring $S = \text{End}_{R}(M)$ of any left $R$–module $R_{M}$ we have a brief observation:

If we have a nonprime endomorphism ring $S$, then there is a nonzero endomorphism $g \in S$ such that $0 \neq \ker g^o \leq M$, vice versa. More precisely, if $S$ is not prime, then there are nonzero endomorphisms $f, g \in S$ such that $fg = 0$. Thus the fact of $fg = 0$ implies that $0 \neq \text{Im}f = Mf \leq \ker g \leq M$. Hence $0 \neq (\ker g)^o \leq M$. Without ambiguity we will write $(\ker g)^o = \ker g^o$ briefly. As a result of the above observation we have the following remark.
Remark 4.1.1. For any module \( R M \), the endomorphism ring \( \text{End}_R(M) \) is not prime if and only if there is a nonzero endomorphism \( g \in \text{End}_R(M) \) such that \( 0 = \ker g \leq M \).

A left \( R \)-module \( R N \) is said to be injective([2], [3], [5]) if for any monomorphism \( i : R K \to R M \) and for any homomorphism \( g : R K \to R N \), there is a homomorphism \( \tilde{g} : R M \to R N \) such that \( i \tilde{g} = g \).

\[
\begin{array}{cccc}
0 & \longrightarrow & R K & \longrightarrow \ R M \\
& & \downarrow g & \swarrow \exists \tilde{g} \\
& & R N & \\
\end{array}
\]

In the above definition of an injective module replacing \( R N \) with \( R M \) we have a definition of a quasi-injective module. Thus it is clear that any injective module is quasi-injective. Therefore the next results are for both quasi-injective modules and injective modules.

**Theorem 4.1.2.** For any (quasi-)injective module \( R N \), if there is an open monomorphism \( f : R M \to R N \), then we have the following.

1. If \( \text{End}_R(M) \) is not a prime ring, then neither \( \text{End}_R(N) \) is.
2. If \( \text{End}_R(N) \) is a prime ring, then so \( \text{End}_R(M) \) is.

**Proof.** (1): Since \( \text{End}_R(M) \) is not a prime ring there are some non-zero non-epic endomorphisms \( g', h' \neq 0 \) in \( \text{End}_R(M) \) such that \( g' h' = 0 \). Therefore there are nonzero endomorphisms \( g, h \in \text{End}_R(f(M)) \) such that \( g \in I^{Mf}, \ h' \in I^{Mf} \) and \( g h = 0 \) (more precisely, let \( g = f^{-1} g' f, \ h = f^{-1} h' f \) and \( g h = 0 \), where \( f^{-1} : R f(M) \to R M \) is the right inverse of a monomorphism \( f \) such that \( f f^{-1} : R M \to R f(M) \to R M \) is the identity mapping on \( R M \) saying that \( \text{End}_R(f(M)) \) is not a prime ring. Hence by the Remark 4.1.1 there is a nonzero endomorphism \( h : R f(M) \to R f(M) \) such that \( 0 = \ker h \leq f(M) \). Since \( R N \) is (quasi-)injective and since \( f(M) \leq N \) there is an extension \( \tilde{h} : R N \to R N \) such that \( h|_{f(M)} = h \) as below.

\[
\begin{array}{cccc}
R M & \overset{g'}{\longrightarrow} & R M & \overset{h'}{\longrightarrow} & R M \\
\downarrow f & & \downarrow f & & \downarrow f \\
R f(M) & \overset{g}{\longrightarrow} & R f(M) & \overset{h}{\longrightarrow} & R f(M) \\
\downarrow & & \downarrow & & \downarrow \\
R N & \overset{\exists \tilde{h}}{\longrightarrow} & R N \\
\end{array}
\]
Since \( f \) is an open monomorphism we also have a nonzero open submodule \( f((\ker h')^o) \leq N \) and hence

\[
\sum_{q \in \text{End}_R(N)} Nq = [f((\ker h')^o)]^o.
\]

Therefore

\[
0 \neq f((\ker h')^o) = \sum_{q \in \text{End}_R(N)} Nq \leq \ker \tilde{h}^o \leq N
\]

for some nonzero endomorphism \( \tilde{h} \in \text{End}_R(N) \). Therefore \( \text{End}_R(N) \) is not prime.

(2): This is the contraposition of (1). \( \square \)

**Corollary 4.1.3.** For an (quasi-)injective module \( R \) \( M \) and a submodule \( K \leq M \), if the inclusion mapping \( \iota : RK \hookrightarrow RM \) is open, then we have the following:

1. If \( \text{End}_R(K) \) is not prime, then neither \( \text{End}_R(M) \) is.
2. If \( \text{End}_R(M) \) is prime, then so \( \text{End}_R(K) \) is.

**Proof.** It is easy to complete the proof by Theorem 4.1.2. \( \square \)

**Corollary 4.1.4.** For an (quasi-)injective self-generated module \( R \) \( M \) and any submodule \( K \leq M \), we have the following.

1. If \( \text{End}_R(K) \) is not prime, then neither \( \text{End}_R(M) \) is.
2. If \( \text{End}_R(M) \) is prime, then so \( \text{End}_R(K) \) is.

**Proof.** Since for a self-generated module \( R \) \( M \) the inclusion mapping \( \iota : RK \hookrightarrow RM \) is always an open monomorphism. Thus the proof is completed by Corollary 4.1.3. \( \square \)

**Examples 4.1.5.** For an injective module \( \mathbb{Z} \) we also have an injective module \( \mathbb{Z} \oplus \mathbb{Z} \leq \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}^{(\infty)} \) has a nonprime endomorphism ring. This fact says that \( \mathbb{Z}^{(\infty)} \) has also a nonprime endomorphism ring.

And an injective non-self-generated module \( \mathbb{Z}[x] \mathbb{Z}[x] \) with a prime endomorphism ring has a submodule \( k\mathbb{Z} + x\mathbb{Z}[x] \leq \mathbb{Z}[x] \) (for \( k \in \mathbb{N} \)) has an open inclusion \( \iota : \mathbb{Z}[x]k\mathbb{Z} + x\mathbb{Z}[x] \hookrightarrow \mathbb{Z}[x] \mathbb{Z}[x] \). It follows from the Corollary 2.3 that the endomorphism ring \( \text{End}_{\mathbb{Z}[x]}(k\mathbb{Z} + x\mathbb{Z}[x]) \) is prime, on the other hand, a
A left $R$-module $R^P$ is said to be projective ([2], [3], [5]) if for any epimorphism $p: R^M \to R^N$ and for any homomorphism $g: R^P \to R^N$, there is a homomorphism $\tilde{g}: R^P \to R^M$ such that $\tilde{g}p = g$.

\[
\begin{array}{ccc}
R^P & \exists \tilde{g} \nearrow & \downarrow g \\
R^M & \xrightarrow{p} & R^N & \xrightarrow{} & 0
\end{array}
\]

In the above definition of a projective module replacing $R^P$ with $R^M$ we have a definition of a quasi-projective module. Thus it is clear that any projective module is quasi-projective. Therefore the next results are for both quasi-projective modules and projective modules.

For any self-generated module $R^M$ and for an open fully invariant submodule $Q \leq M$ of $R^M$, the projection $\pi: R^M \to R^M/Q$ is an open epimorphism with the open preimage assignment of $\pi$.

**Theorem 4.1.6.** For a (quasi-)projective module $R^N$, if there is an $R$-epimorphism $f: R^M \to R^N$ with the open preimage assignment of $f$ and with an open fully invariant submodule $\ker f$, then we have the following.

1. If $\text{End}_R(N)$ is not prime, then neither $\text{End}_R(M)$ nor $\text{End}_R(M/\ker f)$ is.
2. If $\text{End}_R(M)$ is prime, then so $\text{End}_R(N)$ and $\text{End}_R(M/\ker f)$ are.

**Proof.** (1): Suppose that $\text{End}_R(N)$ is not a prime ring. Then there is an endomorphism $g: R^N \to R^N$ such that $0 \neq \ker g \subseteq N$. It is established immediately from the isomorphism theorem that $\text{End}_R(M/\ker f)$ is not a prime ring.

So it remains to show that $\text{End}_R(M)$ is not prime. Since the preimage assignment of $f$ is open we have an open submodule $f^{-1}(\ker g) \leq M$ such that $0 \neq f^{-1}(\ker g) = \sum_{q \in \text{End}_R(M): \ker f \leq f^{-1}(\ker g)\leq M} Mq \leq M$. On the other hand there is the induced isomorphism $\tilde{f}: R^M/\ker f \to R^N$ since $f: R^M \to R^N$ is an epimorphism.

For an endomorphism $\tilde{g} = \tilde{f}g\tilde{f}^{-1}: R^M/\ker f \to R^N \to R^M/\ker f$ since $R^N$ is (quasi-)projective there is an endomorphism $g': R^M/\ker f \to R^M$ such
that \( g'\pi = \tilde{g} \) as in the diagram:

\[
\begin{array}{ccc}
RM/\ker f & \simeq & RN \\
\downarrow & & \\
RM & \xrightarrow{\pi} & RM/\ker f & \rightarrow & 0
\end{array}
\]

Hence we have found an endomorphism \( \pi g' : RM \rightarrow RM/\ker f \rightarrow RM \) such that \( 0 \neq [\ker(\pi g')]^o \leq M \) followed easily from the following commutative diagram:

\[
\begin{array}{ccc}
RN & \xrightarrow{q} & RN & \rightarrow & 0 \\
\downarrow f & & \downarrow f^{-1} & & \\
RM & \xrightarrow{\pi} & RM/\ker f & \xrightarrow{\tilde{g}} & RM/\ker f \\
\downarrow g' & & \downarrow \pi & & \\
RM
\end{array}
\]

Since \( \pi g'\pi = \pi \tilde{g} \) and since the preimage assignment of \( \pi \) is open it follows that \( 0 \neq Mq \leq \pi^{-1}(\ker(g'\pi)^o) = \tilde{\pi}(\ker(g')^o) = \ker(\pi g')^o \leq M \), for some \( 0 \neq q \in \text{End}_R(M) \) which implies that \( 0 \neq \ker(\pi g')^o = \pi^{-1}((\ker g')^o) \leq M \). Therefore \( \text{End}_R(M) \) is not a prime ring.

(2): This is the contraposition of (1). \( \square \)

**Corollary 4.1.7.** For a (quasi-)projective module \( RN \) and for a self-generated module \( RM \), if there is an \( R \)-epimorphism \( f : RM \rightarrow RN \) with a fully invariant kernel \( \ker f \), then we have the following.

1. If \( \text{End}_R(N) \) is not prime, then neither \( \text{End}_R(M/\ker f) \) is.
2. If \( \text{End}_R(M) \) is prime, then so \( \text{End}_R(N) \) and \( \text{End}_R(M/\ker f) \) are.

**Proof.** Since each homomorphism \( f : RM \rightarrow RN \) with a self-generated module \( RM \) has the open preimage assignment and \( \ker f \leq M \) is an open submodule of \( RM \) Theorem 4.1.6 completes the proof. \( \square \)

**Examples 4.1.8.** It is easy to find an epimorphism \( f : \mathbb{Z}(\infty) \rightarrow \mathbb{Z}(2) \) with a fully invariant kernel \( \ker f \) from a self-generated module \( \mathbb{Z}(\infty) \) onto a projective module \( \mathbb{Z}(2) \), where \( \mathbb{Z}(\infty) \) and \( \mathbb{Z}(2) \) are direct sums of infinite and 2-copies of \( \mathbb{Z} \), respectively. It follows immediately from Corollary 4.1.7 that \( \text{End}_\mathbb{Z}(\mathbb{Z}(\infty)) \) is not prime.
4.2. Using kernels of images of endomorphisms

If we have a nonprime endomorphism ring \( S = \text{End}_R(M) \) of a module \( RM \), then there is some nonzero endomorphism \( f \in S \) such that \( 0 \neq \text{Im} f \leq M \), vice versa. More precisely, if \( S \) is not prime, then there are nonzero endomorphisms \( f, g \in S \) such that \( fg = 0 \). Thus the fact of \( fg = 0 \) implies that \( 0 \neq \text{Im} f = Mf \leq \ker g \leq M \). Hence \( 0 \neq \text{Im} f \leq M \).

Remark 4.2.1. For a module \( RM \), the endomorphism ring \( \text{End}_R(M) \) is not prime if and only if there is a nonzero endomorphism \( f \in \text{End}_R(M) \) such that \( 0 \neq Mf \leq M \).

**Theorem 4.2.2.** For an \((\text{quasi-})\)injective module \( RN \), if there is a closed monomorphism \( f : RM \to RN \), then we have the following.

1. If \( \text{End}_R(M) \) is not prime, then neither \( \text{End}_R(f(M)) \) nor \( \text{End}_R(N) \) is.
2. If \( \text{End}_R(N) \) is prime, then \( \text{End}_R(M) \) is prime.

**Proof.** (1): If \( \text{End}_R(M) \) is not a prime ring, then by the isomorphism between \( RM \) and \( RF(M) \) it is clearly obtained that \( \text{End}_R(f(M)) \) is not a prime ring. Thus there is some endomorphism \( g \in \text{End}_R(M) \) such that \( 0 \neq Mg \neq M \). Since \( f \) is closed monomorphism we have a closed submodule \( f(Mg) \leq N \) and \( f(Mg) = \cap_{q \in \text{End}_R(N); f(Mg) \leq \ker q} \ker q \leq N \). Since \( RN \) is \((\text{quasi-})\)injective there is an extension \( \tilde{g} : RN \to RN \) such that \( \tilde{g} | f(M) = f^{-1}gf : Rf(M) \to Rf(M) \) and \( 0 \neq f(Mg) = \cap_{q \in \text{End}_R(N); f(Mg) \leq \ker q} \ker q \leq N \tilde{g} \leq N \), showing that \( \text{End}_R(N) \) is not a prime ring.

(2): This is the contraposition of (1). \( \Box \)

**Corollary 4.2.3.** For any \((\text{quasi-})\)injective self-cogenerated module \( RN \), if there is a monomorphism \( f : RM \to RN \), then we have the following.

1. If \( \text{End}_R(M) \) is not a prime ring. Then neither \( \text{End}_R(N) \) nor \( \text{End}_R(f(M)) \) is prime.
2. If \( \text{End}_R(N) \) is a prime ring. Then so \( \text{End}_R(M) \) and \( \text{End}_R(f(M)) \) are prime.

**Proof.** Since \( RN \) is self-cogenerated any homomorphism \( f : RM \to RN \) is a closed mapping. Theorem 4.2.2 completes the proof. \( \Box \)
Corollary 4.2.4. For any (quasi-)injective self-cogenerated module $R N$ and for any submodule $K \leq R N$, we have the following.

(1) If $\text{End}_R(K)$ is not prime, then neither $\text{End}_R(N)$ is.
(2) If $\text{End}_R(N)$ is prime, then so $\text{End}_R(K)$ is.

Proof. Since $R N$ is self-cogenerated the inclusion mapping $\iota : R K \hookrightarrow R N$ is a closed monomorphism. It follows immediately from Theorem 4.2.2. \qed

Examples 4.2.5. Clearly there is a closed monomorphism $f : \mathbb{Z} \otimes \mathbb{Q}^{(2)} \rightarrow \mathbb{Z} \otimes \mathbb{Q}^{(\infty)}$ from a module $\mathbb{Z} \otimes \mathbb{Q}^{(2)}$ into an injective module $\mathbb{Z} \otimes \mathbb{Q}^{(\infty)}$, where $\mathbb{Z}$ is the integer ring and where $\mathbb{Z} \otimes \mathbb{Q}^{(\infty)}$ and $\mathbb{Z} \otimes \mathbb{Q}^{(2)}$ are direct sums of infinite copies and $2$-copies of the rational field $\mathbb{Q}$, respectively. Thus it follows that the endomorphism ring $\text{End}_\mathbb{Z}(\mathbb{Q}^{(\infty)})$ is not prime from the nonprimeness of $\text{End}_\mathbb{Z}(\mathbb{Q}^{(2)})$.

For any module $R M$ and for a closed fully invariant submodule $Q$ of $R M$, the projection $\pi : R M \rightarrow R M/Q$ is a closed epimorphism with the closed preimage assignment of $\pi$.

Theorem 4.2.6. For a (quasi-)projective module $R N$, if there is a closed epimorphism $f : R M \rightarrow R N$ with the closed preimage assignment and with a closed fully invariant submodule $\ker f \leq M$, then we have the following.

(1) If $\text{End}_R(N)$ is not prime, then neither $\text{End}_R(M)$ nor $\text{End}_R(M/\ker f)$ is.
(2) If $\text{End}_R(M)$ is prime, then so $\text{End}_R(N)$ and $\text{End}_R(M/\ker f)$ are.

Proof. (1): From the nonprime endomorphism ring $\text{End}_R(N)$ it follows that there is a nonzero endomorphism $g : R N \rightarrow R N$ such that $0 \neq \text{Im}g = N \nsubseteq R N$ and $\text{End}_R(M/\ker f)$ is not a prime ring. In other words, there are endomorphisms $g, \phi : R N \rightarrow R N$ such that $0 \neq \text{Im}g \leq \ker f \leq R N$, i.e., $g \phi = 0_{R N}$.

Let $f : R M/\ker f \rightarrow R N$ be the induced isomorphism by $f$. Then we have endomorphisms $\bar{g} = \hat{f} g \hat{f}^{-1}$ and $\bar{\phi} = \hat{f} \phi \hat{f}^{-1} : R M/\ker f \rightarrow R M/\ker f$ such that $0 \neq \text{Im}\bar{g} = (M/\ker f) \bar{g} \leq \ker \bar{\phi} \leq M/\ker f$.

Since $R N \cong R M/\ker f$ is (quasi-)projective there are homomorphisms $g', \phi' : R M/\ker f \rightarrow R M$ and hence there are endomorphisms $k = \pi g', l = \pi \phi' : R M \rightarrow R M$ such that $g' \pi = \bar{g}$ and $\phi' \pi = \bar{\phi}$.
Hence we have found endomorphisms \( \pi g', \pi \phi' : R M \xrightarrow{\pi} R M/\ker f \rightarrow R M \) such that \( 0 \neq \Im(\pi g') \leq \ker \pi \phi' \leq M \) followed easily from the following commutative diagram:

\[
\begin{array}{ccc}
R N & \xrightarrow{g} & R N \\
\downarrow{f} & & \downarrow{f} \\
R M/\ker f & \xrightarrow{\tilde{g}} & R M/\ker f \\
\downarrow{\pi} & & \downarrow{\pi} \\
R M & \xrightarrow{g'} & R M \\
\end{array}
\]

Thus \( \text{End}_R(M) \) is not a prime ring.

(2): This is the contraposition of (1).

**Corollary 4.2.7.** For a (quasi-)projective module \( R N \) and for a self-cogenerated module \( R M \) if there is an epimorphism \( f : R M \rightarrow R N \) with a fully invariant kernel \( \ker f \), then we have the following.

1. If \( \text{End}_R(N) \) is not prime, then neither \( \text{End}_R(M) \) nor \( \text{End}_R(M/\ker f) \) is.
2. If \( \text{End}_R(M) \) is prime, then \( \text{End}_R(N) \) is prime, and thus \( \text{End}_R(M/\ker f) \) is prime.

**Proof.** Since any homomorphism \( f : R M \rightarrow R N \) with a self-cogenerated module \( R M \) is a closed mapping and since the projection \( \pi : R M \rightarrow R M/\ker f \) has the closed image assignment and the closed preimage assignment the proof is established by Theorem 4.2.6.

**Examples 4.2.8.** For a self-cogenerated module \( \mathbb{Z}k \times ( \prod_{n \in \mathbb{N} \setminus k\mathbb{N}} \mathbb{Z}n) \) with any composite number \( k \) and for a projective module \( \mathbb{Z}k \), we have an epimorphism \( f : \mathbb{Z}k \times ( \prod_{n \in \mathbb{N} \setminus k\mathbb{N}} \mathbb{Z}n) \rightarrow \mathbb{Z}k \) such that \( \ker f \) is closed fully invariant.
From the nonprimeness of the endomorphism ring \( \text{End}_\mathbb{Z}(\mathbb{Z}_k) \) it follows that the endomorphism ring \( \text{End}_\mathbb{Z}(\mathbb{Z}_k \times \prod_{n \in \mathbb{N} \setminus \mathbb{N}_0} \mathbb{Z}_n) \) is non-prime.

5. Open meet-prime or closed sum-prime submodules of modules

For fully invariant submodules \( A, B \leq M \), we have that

\[
I_AI_B, I_AI_B \subseteq I_A \cap I_B = I_{A\cap B}
\]

and

\[
I_AI_B, I_BI_A \subseteq I_A \cap I_B = I_{A+B}
\]

hold.

**Lemma 5.1.** For any open \( A \leq M \), open fully invariant \( A_1, A_2, \cdots, A_n \leq M \), and any fully invariant meet-prime submodules \( P, P_1, P_2, \cdots, P_n \leq M \) of a left \( R \)-module \( R \cdot M \) we have the following.

1. If \( A \subseteq \cup_i^nP_i \), then \( A \leq P_i \) for some \( i \).
2. If \( \cap_i^n A_i \leq P \), then \( A_i \leq P \) for some \( i \).
3. If \( \cap_i^n A_i = P \), then \( A_i = P \) for some \( i \).

The following proof is just as the same as the proof of Proposition 1.11 [p. 8, 1].

**Proof.** For fully invariant meet-prime submodules \( P, P_1, P_2, \cdots, P_n \leq M \) we have prime ideals \( I_P, I_{P_1}, I_{P_2}, \cdots, I_{P_n} \leq \text{End}_R(M) \) of the endomorphism ring \( \text{End}_R(M) \) of \( R \cdot M \).

1. By the induction on \( n \) in the form;

\[
A \notin P_i \ (1 \leq i \leq n) \text{ imply that } A \notin \cup_i^n P_i.
\]

For \( n = 1 \), it clearly holds.

For \( n \geq 1 \) we assume that the item (1) is true for \( n - 1 \). Then for each \( i \), there is an endomorphism \( f_i \in I_A \) such that \( f_i \notin I_{P_j} \) for all \( j \neq i \).

If for some \( i \), there is an isomorphism \( f_i \in I_A \) such that \( f_i \notin I_{P_i} \). Then it is proved. If not, there is an isomorphism \( f_i \in I_A \) such that \( f_i \notin I_{P_i} \) for all \( i \).

Considering an endomorphism \( g = \sum_{i=1}^n f_1 f_2 \cdots f_{i-1} f_{i+1} \cdots f_n \notin \prod_{i \neq i} P_i \).

Then we have that \( M \cdot g \leq A \) but \( M \cdot g \notin \cup_i^n P_i \). From the openness of \( A \) it follows that \( A \leq P_i \) for some \( i \). Therefore the item (1) is true.

2. Suppose that \( P \notin A_i \) for every \( i \leq i \leq n \). Then there is some endomorphism \( f_i \in I_A \) such that \( f_i \notin I_P \) for every \( i \). And hence \( g = \prod_i^n f_i \in \text{End}_R(M) \).
\[ \prod_i I^{A_i} \subseteq \cap_i I^{A_i} \setminus I_P = I^{\cap_i A_i} \setminus I_P \] since \( I_P \) is prime. Then it concludes that \( P \not\subseteq \cap_i A_i \).

(3): If \( P = \cap_i A_i \), then from the above (2) it follows immediately that \( P = A_i \) for some \( i \). \qed

**Lemma 5.2.** For any closed submodule \( B \leq M \), any closed fully invariant submodules \( B_1, B_2, \cdots, B_n \leq M \), and any fully invariant sum-prime submodules \( Q, Q_1, Q_2, \cdots, Q_n \leq M \) of any \( R \)-module \( R M \), we have the following.

1. If \( B \supseteq \cup_i Q_i \), then \( B \supseteq Q_i \) for some \( i \).
2. If \( Q \subseteq \sum_i B_i \), then \( Q \subseteq B_i \) for some \( i \).
3. If \( Q = \sum_i B_i \), then \( B_i = Q \) for some \( i \).

**Proof.** For fully invariant meet-prime submodules \( Q_1, Q_2, \cdots, Q_n \leq M \) we have prime ideals \( I_{Q_1}, I_{Q_2}, \cdots, I_{Q_n} \subseteq \text{End}_R(M) \) of the endomorphism ring \( \text{End}_R(M) \).

(1): By the induction on \( n \) in the form;

\[ B \not\supseteq Q_i \ (1 \leq i \leq n) \] imply that \( B \not\supseteq \cup_i Q_i \).

For \( n = 1 \), it clearly holds.

For \( n \geq 1 \) we assume that the item (1) is true for \( n - 1 \). Then for each \( i \), there is an endomorphism \( f_i \in I_B \) such that \( f_i \not\in I_{Q_i} \) for all \( j \neq i \).

If for some \( i \), there is an isomorphism \( f_i \in I_B \) such that \( f_i \not\in I_{Q_i} \). Then it is proved. If not, there is an isomorphism \( f_i \in I_B \) such that \( f_i \not\in I_{Q_i} \) for all \( i \). Considering an endomorphism \( g = \sum_{i=1}^n f_1 f_2 \cdots f_{i-1} f_{i+1} \cdots f_n \not\in \cup_i Q_i \). Then we have that \( \ker g \supseteq B \) but \( \ker g \not\supseteq \cup_i Q_i \). From the closedness of \( B \) it follows that \( B \supseteq Q_i \) for some \( i \). Therefore the item (1) is true.

(2): Suppose that \( Q \not\subseteq B_i \) for every \( i \) \( (1 \leq i \leq n) \). Then there is some endomorphism \( f_i \in I_B \) such that \( f_i \not\in I_Q \) for every \( i \). And hence \( g = \prod_i f_i \in \prod_i I_{B_i} \subseteq \cap_i I_{B_i} \setminus I_Q = I_{\sum_i B_i} \setminus I_Q \) since \( I_Q \) is prime. Then it concludes that \( Q \not\subseteq \sum_i B_i \).

(3): If \( Q = \sum_i B_i \), then from the above (2) it follows immediately that \( Q = B_i \) for some \( i \). \qed

**Remark 5.3.** Any maximal submodule \( N \leq M \) of a module \( R M \) (if \( R M \) has any) is meet-prime and any minimal submodule (if \( R M \) has any) is sum-prime.
Proposition 5.4. For any module \( R M \), we have the following.

1. There exists at least one proper maximal open submodule (that is, maximal submodule among the open submodules) of \( R M \).

2. There exists at least one nonzero minimal closed submodule (that is, minimal submodule among the closed submodules) of \( R M \).

Proof. (1): Let \( \mathcal{S} = \{ A \leq M | A \text{ is a proper open submodule of } R M \} \) be the set of all proper open submodules of \( R M \). Then \( \mathcal{S} \neq \emptyset \) since the trivial submodule 0 is open. Let \( \mathcal{C} \) be any chain in \( \mathcal{S} \) of proper open submodules of \( R M \). Then \( \mathcal{C} : \cdots \leq A_1 \leq A_2 \leq \cdots \leq A_n \leq \cdots \) has an upper bound \( \cup A_i \) which is an open submodule of \( R M \). By the Zorn’s lemma there exists a maximal element \( \cup A_i = A \leq M \) in \( \mathcal{S} \), in fact, which is a maximal among proper open submodules of \( R M \).

Easily it follows from Definition 1.1 that such a maximal element \( A \) is a meet-prime submodule of \( R M \).

(2): Let \( \mathcal{T} = \{ B(\neq 0) \leq M | B \text{ is a nonzero closed submodule of } R M \} \) be the set of all nonzero closed submodules of \( R M \). Then \( \mathcal{T} \neq \emptyset \) since the trivial submodule 0 is closed. Let \( \mathcal{D} \) be any chain in \( \mathcal{T} \) of nonzero closed submodules of \( R M \). Then \( \mathcal{D} : \cdots \geq B_1 \geq B_2 \geq \cdots \geq B_n \geq B_{n+1} \geq \cdots \) has a lower bound \( \cap B_i \) which is a closed submodule of \( R M \). By the Zorn’s lemma with a reversing set inclusion order there exists a minimal element \( \cap B_i = B \leq M \) in \( \mathcal{T} \).

Easily it follows from Definition 1.7 that such a minimal element \( B \) is a sum-prime submodule of \( R M \). \( \square \)

Remark 5.5. In spite of the Proposition 5.4 it is not guaranteed for the sets

\[ \{ P \leq M | P \text{ is a proper fully invariant meet-prime submodule of } R M \} \]

and

\[ \{ P \neq 0 | P \text{ is a nonzero fully invariant sum-prime submodule of } R M \} \]

(which will be studied in the sections 7 and 8) to be nonempty sets, for any module \( R M \).

6. Zariski topologies for endomorphism rings

It is trivial that if an endomorphism ring \( S \) has no prime ideal of \( S \), then \( S \) is not prime.
For any left module $R \cdot M$ over a ring $R$, there exists a proper fully invariant meet-prime or proper fully invariant sum-prime submodule $P$, respectively, we have a prime ideal $I^P$ or $I_P$ in the endomorphism ring $S = \text{End}_R(M)$. Unfortunately this does not guarantee the existence of a proper prime ideal of $S$.

We let $\text{Spec}(S)$ be the set of all prime ideals of $S$ (even though $S$ need not to be a commutative ring), precisely

$$\text{Spec}(S) = \{ J \triangleleft S \mid J \text{ is a prime ideal of } S \}$$

which will be called the prime spectrum of the endomorphism ring $S$. Then we also have a topological space which will be named by Zariski topology on the spectrum $\text{Spec}(S)$ as follows:

**Theorem 6.1.** For any module $R \cdot M$, the prime spectrum $\text{Spec}(S)$ of the endomorphism ring $S$ is a topological space, if as closed sets we take all sets of form $v(E) = \{ I \in \text{Spec}(S) \mid E \subseteq I \}$, where $E$ is any subset of $S$. Precisely, the sets $v(E)$ satisfy the axioms for closed sets in a topological space.

1. For any subset $E \subseteq S$, if $(E)$ is the ideal of $S$ generated by $E$, then $v(E) = v((E)) = v(r(E))$, where $r(E) = \cap_{E \subseteq J, J \in \text{Spec}(S)} J$ is the prime radical of $E$.
2. $v(0) = \text{Spec}(S)$, $v(S) = \emptyset$.
3. $v(\bigcup_{i \in I} E_i) = \cap_{i \in I} v(E_i)$, for each $E_i \subseteq S$.
4. $v(AB) = v(A) \cup v(B)$ for $A, B \subseteq S$.

**Proposition 6.2.** For any left $R$-module $R \cdot M$, $\text{Spec}(S)$ is a topological space, if as open sets we take all sets of form

$$\Gamma A = \{ J \in \text{Spec}(S) \mid A \nsubseteq J \},$$

where $A$ is any subset of $S$.

Before a proof, it is convenient to note that

$$\Gamma A = \{ J \in \text{Spec}(S) \mid A \nsubseteq J \} = \{ J \in \text{Spec}(S) \mid \langle A \rangle \nsubseteq J \},$$

for $A$ is any subset of $S$, where $\langle A \rangle$ is the ideal generated by the set $A$.

Additionally notice that for any subset $A$ of $S$

$$\Gamma A = \Gamma(\sum_{a \in A} \langle a \rangle) = \cap_{a \in A} \Gamma a = \cap_{a \in A} \Gamma \langle a \rangle$$

$$= \{ J \in \text{Spec}(S) \mid A \nsubseteq J \} = \{ J \in \text{Spec}(S) \mid \langle A \rangle \nsubseteq J \}$$

$$= \Gamma(\cap_{A \nsubseteq J, J \beta} J_\beta), \quad J_\beta \text{ is a prime ideal of } S.$$
The resulting topology is called the Zariski topology named after the Zariski topology on the prime spectrum of a commutative ring. The topological space Spec($S$) is called the prime spectrum of the endomorphism ring $S$ of a module $R_M$.

Remind that a topological space $X$ is said to be irreducible if $X \neq \emptyset$ and if every nonempty two open sets intersect, or equivalently if every nonempty open set is dense in $X$ (p. 13 in [1]).

**Theorem 6.3.** For any module $R_M$, the following are equivalent:

1. Spec($S$) is irreducible;
2. The prime radical $\text{rad}(S) = \cap_{J \in \text{Spec}(S)} J$ is in Spec($S$), i.e., $\text{rad}(S)$ is a prime ideal of $S$.

### 7. Zariski image topologies for openly regular modules

A module $R_M$ is said to be openly regular if for any submodules $C, D \leq M$, the following properties are satisfied:

1. $C^o \leq D^o$ implies that $C \leq D$,
2. $C^o = D^o$ implies that $C \leq D$ or $D \leq C$.

Clearly any self-generated module is openly regular. There are openly regular modules which are not self-generated, for instance, a module $\mathbb{Z}_p[x]$ for the polynomial ring $\mathbb{Z}_p[x]$ over the ring $\mathbb{Z}_p$ modulo $p$ has nonopen submodules $x^n\mathbb{Z}_p[x] \leq \mathbb{Z}_p[x] \mathbb{Z}_p[x]$ ($n \in \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers) having the trivial submodule $0 = (x^n\mathbb{Z}_p[x])^o \leq x^n\mathbb{Z}_p[x]$ ($n \in \mathbb{N}$). Clearly it is seen that $\{x^n\mathbb{Z}_p[x] : n \in \mathbb{N}\}$ is linearly ordered. We have the following results relative to meet-prime submodules of left $R$-modules:

Let

$$\Pi = \{P_\alpha \leq M | P_\alpha \text{ is a proper fully invariant meet-prime submodule of } R_M\}$$

be the set of all proper fully invariant meet-prime submodules of $R_M$. Then we have the following proposition.

**Proposition 7.1.** For any openly regular left $R$–module $R_M$, $\Pi$ is a topological space, if as closed sets we take all sets of form $v(E) = \{P \in \Pi \mid E \subseteq P\}$, where $E$ is any subset of $R_M$. Precisely, the sets $v(E)$ satisfy the axioms for closed sets in a topological space:

1. For any subset $E \subseteq M$, if $\langle E \rangle$ is the submodule of $M$ generated by $E$, then $v(E) = v(\langle E \rangle) = v(r(E))$, where $r(E) = \cap_{E \subseteq P_\alpha \in \Pi} P_\alpha$ is the prime radical of $E$.
2. $v(0) = v(r(0)) = \Pi$, $v(M) = \emptyset$. 
(3) \( v(\cup_{i \in I} E_i) = \cap_{i \in I} v(E_i) \), for each \( E_i \subseteq M \).

(4) \( v(A \cap B) = v(A) \cup v(B) \) for \( A, B \subseteq M \).

The prime radical \( \text{rad}(M) = r(0) = \cap_{P \in \Pi} P \) of any \( R \) is an open fully invariant submodule of \( R \).

Proof. (4): If \( A \cap B \subseteq P \) for \( P \in \Pi \), then \( \langle A \rangle^{o} \cap \langle B \rangle^{o} \subseteq P^{o} \) implies that \( \langle A \rangle^{o} \leq P^{o} \) or \( \langle B \rangle^{o} \leq P^{o} \) since \( P \) is meet-prime if and only if \( P^{o} \) is meet-prime. Then it follows that \( A \subseteq \langle A \rangle \leq P \) or \( B \subseteq \langle B \rangle \leq P \) by letting \( A = B \) in (*)).

(*) If \( A \cap B \subseteq P \), then \( \langle A \rangle^{o} \cap \langle B \rangle^{o} \leq P^{o} \iff \langle A \rangle \cap \langle B \rangle \leq P \) for any meet-prime \( P \leq M \) in any openly regular module \( R \). In order to show (*), suppose that \( \langle A \rangle \geq P \) and \( \langle B \rangle \geq P \). Then \( A^{o} \cap B^{o} = P^{o} \) follows and hence \( \langle A \rangle^{o} = \langle B \rangle^{o} = (\langle A \rangle \cap \langle B \rangle)^{o} = P^{o} \) is fully invariant meet-prime. Hence \( P^{o} \leq \langle A \rangle \cap \langle B \rangle \). Since \( R \) is openly regular we have that \( \langle A \rangle \cap \langle B \rangle \cap \langle A \rangle \cap \langle B \rangle \) and \( P \) are submodules of \( R \) which are linearly ordered. Thus \( P \subset \langle A \rangle \cap \langle B \rangle = \langle A \cap B \rangle \subset \langle A \rangle \cap \langle B \rangle \) (which is contradicted to \( A \cap B \not\subseteq P \)) or \( \langle A \rangle \cap \langle B \rangle \not\subseteq P \subset \langle A \rangle \cap \langle B \rangle \) (which is the required) follows. Hence the only case of \( \langle A \rangle \cap \langle B \rangle \not\subseteq P \subset \langle A \rangle \cap \langle B \rangle \) remains to be considered, and hence we have that \( A \cap B \subseteq P \). Therefore if \( A \cap B \subseteq P \), we have that \( \langle A \rangle^{o} \leq P \iff \langle A \rangle \cap \langle B \rangle \not\subseteq P \) for any meet-prime \( P \leq M \) in any openly regular module \( R \). Conversely, \( v(A) \cup v(B) \subseteq v(A \cap B) \) is elementary. Therefore we have proved (4).

\[
\text{PROPOSITION 7.2. For any openly regular left } R \text{-module } R, \text{ \Pi is a topological space, if as open sets we take all sets of form } \\
\Gamma A = \{ P \in \Pi \mid A \not\subseteq P \},
\]

where \( A \) is any subset of \( R \).

It is convenient to note that

\[
\Gamma A = \{ P \in \Pi \mid A \not\subseteq P \} = \{ P \in \Pi \mid \langle A \rangle \not\subseteq P \},
\]

for \( A \) is any subset of \( R \), where \( \langle A \rangle \) is the submodule generated by the set \( A \).

Additionally notice that for any subset \( A \subseteq M \) of \( R \),

\[
\Gamma A = \Gamma(\sum_{a \in A} (a)) = \cap_{a \in A} \Gamma a = \cap_{a \in A} \Gamma(a) \\
= \{ P \in \Pi \mid A \not\subseteq P \} = \{ P \in \Pi \mid \langle A \rangle \not\subseteq P \} \\
= \Gamma(\cap_{\alpha \in P(a)} P_{\alpha}),
\]

where \( P_{\alpha} \) is a fully invariant meet-prime submodule of \( R \).
The resulting topology is called the Zariski image topology for the openly regular $RM$ named after the Zariski topology on the prime spectrum of a commutative ring. The topological space $\Pi$ is called the image spectrum of $RM$, denoted by $\text{Spec}_I(M)$.

Also we define the prime radical $\text{rad}(M)$ by the intersection of all meet-prime submodules of $RM$, in other words, $\text{rad}(M) = \cap_\alpha P_\alpha$ (cf. the Jacobson Radical $\text{Rad}(M)$ the intersection of all maximal submodules of $RM$).

Clearly in any openly regular module $RM$ it is easily shown that $\text{rad}(M) \leq \text{Rad}(M)$ (if $\text{Rad}(M) \neq M$ i.e., if $RM$ has any maximal submodule of $RM$).

Let $\mathfrak{S}$ be the set of all open submodules of $RM$, then by the Zorn’s lemma there are maximal submodules among open submodules of $RM$, being open fully invariant meet-prime submodules of $RM$. This says that $\text{Spec}_I(M)$ is a nonempty set.

If the prime radical $\text{rad}(M)$ is a meet-prime submodule of $RM$, then the image spectrum $\text{Spec}_I(M) = \{L \leq M \mid \text{rad}(M) \leq L\}$ contains $\text{rad}(M)$ since the prime radical $\text{rad}(M)$ is open and fully invariant in $RM$.

Theorem 7.3. For any openly regular module $RM$, if a submodule $K \leq \text{rad}(M)$ of $RM$ is in $\text{Spec}_I(M)$, then we have that $K = \text{rad}(M)$ and $\text{Spec}_I(M)$ is irreducible.

Proof. If $K \in \text{Spec}_I(M)$, then $K$ is fully invariant meet-prime, then the open submodule $K^\circ$ is also fully invariant meet-prime in $RM$. Thus $\text{rad}(M) \leq K^\circ \in \text{Spec}_I(M)$ implies that $\text{rad}(M) = K \in \text{Spec}_I(M)$.

And every basic open set in the image spectrum $\text{Spec}_I(M)$ contains $\text{rad}(M)$, in other words, $\text{Spec}_I(M)$ is irreducible. And by the hypothesis of $K \leq \text{rad}(M)$, we have an open submodule $\text{rad}(M) = K^\circ$ which is in $\text{Spec}(M)$. □

Corollary 7.4. For any openly regular module $RM$, we have that $\text{Spec}_I(M)$ is irreducible if and only if $\text{rad}(M) \in \text{Spec}_I(M)$.

For any module $RM$, we have a surjective mapping from the image spectrum $\text{Spec}_I(M)$ onto a subset $\{I^P \mid P \in \text{Spec}_I(M)\} \subseteq \text{Spec}(S)$ of the prime spectrum $\text{Spec}(S)$ of the endomorphism ring $S$ of $RM$. Let this subspace $\{I^P \mid P \in \text{Spec}_I(M)\}$ be the topological subspace of the Zariski topology of the spectrum $\text{Spec}(S)$ of the endomorphism ring. Then we have the next theorem.
Lemma 7.5. For any openly regular module $R M$, let
\[ Y = \{ I^P | P \in \text{Spec}_I(M) \} \subseteq \text{Spec}(S). \]

Then we have the following.

(1) If $Y$ is open in $\text{Spec}(S)$ and if the prime spectrum $\text{Spec}(S)$ is irreducible, then the image spectrum $\text{Spec}_I(M)$ is irreducible.

(2) If $Y$ is dense in $\text{Spec}(S)$ and if the image spectrum $\text{Spec}_I(M)$ is irreducible, then the prime spectrum $\text{Spec}(S)$ is irreducible.

(3) If $Y$ is open dense in $\text{Spec}(S)$, then the prime spectrum $\text{Spec}(S)$ is irreducible if and only if the image spectrum $\text{Spec}_I(M)$ is irreducible.

Corollary 7.6. For any openly regular module $R M$, if
\[ \{ I^P | P \in \text{Spec}_I(M) \} \]

is open dense in $\text{Spec}(S)$, then the following are equivalent:

(1) The prime spectrum $\text{Spec}(S)$ is reducible;

(2) The image spectrum $\text{Spec}_I(M)$ is reducible.

Remark 7.7. The openness and density of $\{ I^P | P \in \text{Spec}_I(M) \}$ in the hypotheses of the Proposition 7.5 and Corollary 7.6 is essential. Without the openness of the subspace $Y$, it is impossible for $Y$ to contain the prime radical of $S$. For example, a module $Z$ over the integer ring $\mathbb{Z}$ has a non-open prime image spectrum $\text{Spec}_I(M)$ isomorphic to $\{ p\mathbb{Z} | p \text{ is a prime number} \}$ but its prime radical $\text{rad}(Z) = 0 \notin \text{Spec}_I(\mathbb{Z})$, in other words, $Y = \{ p\mathbb{Z} | p \text{ is a prime number} \}$ is not open in $\text{Spec}(S)$. However it is well-known that the prime spectrum $\text{Spec}(\text{End}_{\mathbb{Z}}(\mathbb{Z}))$ is irreducible. And for a prime number $p$ considering a left $\mathbb{Z}(p^\infty)$ having an empty set $Y = \{ p\mathbb{Z} | p \text{ is a meet-prime submodule of } \mathbb{Z}(p^\infty) \} = \emptyset \subseteq \text{Spec}(\text{End}_{\mathbb{Z}}(\mathbb{Z}(p^\infty)))$, then we have that $Y$ is reducible and $\text{Spec}(\text{End}_{\mathbb{Z}}(\mathbb{Z}(p^\infty)))$ is a singleton being irreducible in the Zariski topology. This shows that the reducibility of $Y$ does not imply that of $\text{Spec}(S)$ without the density of $Y$.

Considering the quotient module $R M/\text{rad}(M)$ of any module $R M$ over the prime radical $\text{rad}(M)$ of module $R M$, let $T = \text{End}_R(M/\text{rad}(M))$ denote the endomorphism ring of the quotient module $R M/\text{rad}(M)$ over the prime radical $\text{rad}(M)$.
Theorem 7.8. For an openly regular module $R M$ with the prime radical $\text{rad}(M)$, if $\{I^L | L \in \text{Spec}_I(M/\text{rad}(M))\}$ is open dense in $\text{Spec}(T)$, where $T = \text{End}_R(M/\text{rad}(M))$ is the endomorphism ring of the quotient module $R M/\text{rad}(M)$, the following are equivalent:

(1) The endomorphism ring $\text{End}_R(M/\text{rad}(M))$ is prime;
(2) The prime spectrum $\text{Spec}(T)$ is irreducible;
(3) The image spectrum $\text{Spec}_I(M/\text{rad}(M))$ is irreducible;
(4) The prime radical $\text{rad}(M)$ of $R M$ is meet-prime.

Proof. (1) $\implies$ (2): It is trivial.
(2) $\implies$ (3): Assume (1), then the prime spectrum $\text{Spec}(T)$ is irreducible. Thus by the above Lemma 7.4 we have the irreducible image spectrum $\text{Spec}_I(M/\text{rad}(M))$.
(3) $\implies$ (4): From Corollary 7.4 it follows immediately.
(4) $\implies$ (1): Assume that the image spectrum $\text{Spec}_I(M/\text{rad}(M))$ is irreducible, then the prime radical $\text{rad}(M/\text{rad}(M)) = \text{rad}(M) \in \text{Spec}_I(M/\text{rad}(M))$ is a fully invariant meet-prime submodule of an openly regular module $R M/\text{rad}(M)$. Therefore we obtain a prime ideal $I^{\text{rad}(M)} = I^T = 0 \subseteq T$ of the endomorphism ring of $R M/\text{rad}(M)$. Therefore the endomorphism ring $T$ is a prime ring. □

Theorem 7.9. For an openly regular module $R M$ with the prime radical $\text{rad}(M)$, if $\{I^P | P \in \text{Spec}_I(M)\}$ is open dense in $\text{Spec}(T)$, where $T = \text{End}_R(M/\text{rad}(M))$ is the endomorphism ring of the quotient module $R M/\text{rad}(M)$, the following are equivalent:

(1) The endomorphism ring $\text{End}_R(M/\text{rad}(M))$ is not prime;
(2) The prime spectrum $\text{Spec}(T)$ is reducible;
(3) The image spectrum $\text{Spec}_I(M/\text{rad}(M))$ is reducible;
(4) The prime radical $\text{rad}(M)$ of $R M$ is not meet-prime.

Theorem 7.10. For an openly regular module $R M$ with $\text{rad}(M) = 0$, if $\{I^P | P \in \text{Spec}_I(M)\}$ is open dense in $\text{Spec}(S)$, then the following are equivalent:

(1) The endomorphism ring $S$ is prime;
(2) The prime spectrum $\text{Spec}(S)$ is irreducible;
(3) The image spectrum $\text{Spec}_I(M)$ is irreducible;
(4) 0 is meet-prime.

Proof. Replacing $\text{rad}(M)$ with 0 in the above Theorem 7.7, the proof is completed. □
Theorem 7.11. For an openly regular module $R M$ with $\text{rad}(M) = 0$, if \{\overline{P} | P \in \text{Spec}(M)\} is open dense in $\text{Spec}(S)$, then the following are equivalent:

1. The endomorphism ring $S$ is not prime;
2. The prime spectrum $\text{Spec}(S)$ is reducible;
3. The image spectrum $\text{Spec}_I(M)$ is reducible;
4. 0 is not meet-prime.

8. Zariski kernel(null) topologies for closedly regular modules

A module $R M$ is said to be closedly regular if for any submodules $C, D \leq M$, the following properties are satisfied:

1. $C \leq D$ implies that $C \subseteq D$,
2. $C = D$ implies that $C \leq D$ or $D \leq C$.

Clearly any self-cogenerated module is closedly regular. There are closedly regular modules which are not self-cogenerated, for example, a closedly regular $\mathbb{Z}[x]$-module $\mathbb{Z}(p^\infty)[x]$ has non-closed submodules $x^n\mathbb{Z}(p^\infty)[x]$ ($n \in \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers) including the trivial submodule $\mathbb{Z}(p^\infty)[x] = x^n\mathbb{Z}(p^\infty)[x]$. Also \{ $x^n\mathbb{Z}(p^\infty)[x] | n \in \mathbb{N}$\} is linearly ordered.

Let $\mathcal{S}$ be the set of all closed submodules of $R M$ with a reversing order of set inclusion, then by the Zorn’s lemma there are maximal submodules among closed submodules of $R M$, being closed fully invariant sum-prime submodules of $R M$. Thus it follows that

$$\mathcal{S} = \{ Q \leq M | Q \text{ is a sum-prime submodule of } R M \} \neq \emptyset$$

but

$$\{ Q \leq M | 0 \neq Q \text{ is a nonzero sum-prime submodule of } R M \} \neq \emptyset$$

is not held, in general. With a risk of being empty set, we will introduce a topological space on the set of all nonzero fully invariant sum-prime submodules of any closedly regular module over any ring as follows.

Let $\Xi = \{ P_n \neq 0 | P_n \text{ is a nonzero fully invariant sum-prime submodule of } R M \}$ be the set of all non-zero fully invariant sum-prime submodules of $R M$. Then we have the following proposition.

Proposition 8.1. For a closedly regular left $R$-module $R M$, $\Xi$ is a topological space, if as closed sets we take all sets of form

$$w(E) = \{ P \in \Xi | P \subseteq E \},$$
where $E \subseteq M$ is any subset of $R.M$. Precisely, the sets $w(E)$ satisfy the axioms for closed sets in a topological space:

1. For any subset $E \subseteq M$, if $(E)$ is the submodule of $M$ generated by $E$, then $w(E) = w((E)) = w(\text{soc}(E))$, where $\text{soc}(E) = \sum_{P_\alpha \in \Xi} P_\alpha$ is the prime socle of $E$.
2. $w(M) = w(\text{soc}(M)) = \Xi$, $w(0) = \emptyset$.
3. $w(\cap_{i \in I} E_i) = \cap_{i \in I} w(E_i)$ for $E_i \subseteq M (i \in I)$.
4. $w(A \cup B) = w((A) + (B)) = w(A) \cup w(B)$ for $A, B \subseteq M$.

**Proof.** (4): Trivially it is true that $w((A) + (B)) = w(A) \cup w(B) \subseteq w(A \cup B)$.

It remains to show that $w(A \cup B) \subseteq w((A) + (B)) = w(A) \cup w(B)$. Let $P$ be any sum-prime submodule of $R.M$ such that $P \leq A \cup B$, then $P \leq (A) + (B) = (A) + (B)$ and then $P \leq (A)$ or $P \leq (B)$ by (2) of the Lemma 5.2. Since $R.M$ is closedly regular and since $P \leq (A) + (B) \iff P \leq (A) + (B)$ we have that $P \leq A$ or $P \leq B$ (otherwise if $P \geq (A)$ and if $P \geq (B)$, then $P \geq (A) + (B) = (A \cup B)$ and it is contradicted to $P \leq A \cup B$.) Thus we have $w(A \cup B) \subseteq w((A) + (B)) = w(A) \cup w(B)$.

**Proposition 8.2.** $\Xi$ is a topological space, if as open sets we take all sets of form $\tau A = \{ P \in \Xi | P \nsubseteq A \}$, where $A \subseteq M$ is any subset of $r.M$.

Before a proof, it is convenient to note that

$\tau A = \{ P \in \Xi | P \nsubseteq A \} = \{ P \in \Xi | P \nsubseteq (A) \}$,

where $A$ is any subset of $M$ and $(A)$ is the submodule of $r.M$ generated by the set $A$. Additionally notice that for any subset $A$ of $R.M$

$\tau A = \cup_{a \in A} \tau a$

$= \cup_{a \in A} \tau(a)$

$= \tau(\sum_{a \in A} (a))$

$= \{ P \in \Xi | P \nsubseteq A \}$

$= \{ P \in \Xi | P \nsubseteq (A) \}$

$= \tau(\cap_{P_\beta \nsubseteq A} P_\beta)$,

for which $P_\beta$ is a non-zero closed fully invariant sum-prime submodule of $R.M$. 
The resulting topology is called the \textit{Zariski kernel} (or \textit{null}) topology for $R M$ named after the Zariski topology on the prime spectrum of a commutative ring. The topological space $\Xi$ is called the \textit{kernel} (or \textit{null}) topology of $M$, denoted by $\text{Spec}_N(M)$. Also we define the \textit{prime socle} $\text{soc}(M)$ by the sum of all sum-prime submodules of $R M$, in other words, $\text{soc}(M) = \sum_{P_{\alpha} \in \Xi} P_{\alpha}$ (cf. the \textit{Socle} $\text{Soc}(M)$ the sum of all minimal submodules of $R M$). Clearly in any closedly regular module it follows easily that $\text{soc}(M) \leq \text{Soc}(M)$.

If the prime socle $\text{soc}(M)$ is a sum-prime submodule of $R M$, then $\text{Spec}_N(M) = \{ L \neq 0 \mid L \leq \text{soc}(M) \}$ contains $\text{soc}(M)$ since the prime radical $\text{soc}(M)$ is closed and fully invariant in $R M$.

**Theorem 8.3.** For any closedly regular module $R M$, if a submodule $K \geq \text{soc}(M)$ of $R M$ is in $\text{Spec}_N(M)$, then we have that $K = \text{soc}(M)$ and $\text{Spec}_N(M)$ is irreducible.

**Proof.** If $K \in \text{Spec}_N(M)$, then $K$ is fully invariant sum-prime, then the closed submodule $\overline{K}$ is also fully invariant sum-prime in $R M$. Thus $\text{soc}(M) \leq K \leq \overline{K} \in \text{Spec}_N(M)$ implies that $\text{soc}(M) = \overline{K} = K \in \text{Spec}_N(M)$. And every basic open set in the kernel(null) spectrum $\text{Spec}_N(M)$ contains $\text{soc}(M)$, in other words, $\text{Spec}_N(M)$ is irreducible. And by the hypothesis of $K \geq \text{soc}(M)$, we have a closed submodule $\text{soc}(M) = K$ which is in $\text{Spec}(M)$. \hfill \Box

**Corollary 8.4.** For any closedly regular module $R M$, the following are equivalent:

1. $\text{Spec}_N(M)$ is irreducible;
2. $\text{soc}(M) \in \text{Spec}_N(M)$.

For any module $R M$, we have a surjective mapping from the kernel(null) spectrum $\text{Spec}_N(M)$ onto a subset

$$\{I_P \mid P \in \text{Spec}_N(M)\} \subseteq \text{Spec}(S)$$

of the prime spectrum $\text{Spec}(S)$ of the endomorphism ring $S$ of $R M$. Let this subspace $\{I_P \mid P \in \text{Spec}_N(M)\}$ be a topological subspace of the Zariski topology of the spectrum $\text{Spec}(S)$ of the endomorphism ring. Then we have the next theorem.
Lemma 8.5. For any closedly regular module \( R \) let
\[
Y = \{ I_P | P \in \text{Spec}_N(M) \} \subseteq \text{Spec}(S),
\]
then we have the following.

1. If \( Y \) is open in \( \text{Spec}(S) \) and if the prime spectrum \( \text{Spec}(S) \) is irreducible, then the kernel(null) spectrum \( \text{Spec}_N(M) \) is irreducible.

2. If \( Y \) is dense in \( \text{Spec}(S) \) and if the kernel(null) spectrum \( \text{Spec}_N(M) \) is irreducible, then the prime spectrum \( \text{Spec}(S) \) is irreducible.

3. If \( Y \) is open dense in \( \text{Spec}(S) \). Then the prime spectrum \( \text{Spec}(S) \) is irreducible if and only if the kernel(null) spectrum \( \text{Spec}_N(M) \) is irreducible.

Proof. (1): By the hypothesis of irreducibility of \( \text{Spec}(S) \), it follows that its subspace is irreducible since the closure of an open set in the subspace \( \{ I_P | P \in \text{Spec}_N(M) \} \) is the intersection of the closure of the open set in \( \text{Spec}(S) \) and the subspace \( \{ I_P | P \in \text{Spec}_N(M) \} \) is inherited from the Zariski topology. The Zariski kernel topology \( \text{Spec}_N(M) \) is the same that the topology with an onto mapping \( P \mapsto I_P : \text{Spec}_N(M) \to Y \) satisfies that each basic open set contains preimage of a basic open set in \( Y = \{ I_P | P \in \text{Spec}_N(M) \} \). Therefore \( \text{Spec}_N(M) \) is also irreducible.

(2): Assume that the prime spectrum \( \text{Spec}(S) \) is reducible. Then there are two nonempty disjoint open subsets in \( \text{Spec}(S) \) inducing two disjoint nonempty open subsets in \( Y \) since \( Y \) is dense in \( \text{Spec}(S) \). Therefore it follows easily that \( \text{Spec}_N(M) \) is reducible.

(3): From (1) and (2) it follows immediately. \( \square \)

Corollary 8.6. For any openly regular module \( R \) \( M \), if
\[
\{ I_P | P \in \text{Spec}_N(M) \}
\]
is open dense in \( \text{Spec}(S) \), then the following are equivalent:

1. The prime spectrum \( \text{Spec}(S) \) is reducible;
2. The kernel(null) spectrum \( \text{Spec}_N(M) \) is reducible.

Remark 8.7. The openness and density of \( \{ I_P | P \in \text{Spec}_N(M) \} \) in the hypotheses of the Proposition 8.5 and Corollary 8.6 is essential. For example, a \( \mathbb{Z} \)–module \( \mathbb{Z}(p^\infty) \) for a prime number \( p \) has a non-sum-prime submodule \( \text{soc}(\mathbb{Z}(p^\infty)) = \mathbb{Z}(p^\infty) \notin \text{Spec}_N(\mathbb{Z}(p^\infty)) \), in other words, \( \{ I_K | K \text{ is a nonzero} \).
fully invariant sum-prime submodule of $\mathbb{Z}(p^\infty)$ is not an open set in the prime spectrum Spec$(S) \ni 0 = I_{\text{soc}(\mathbb{Z}(p^\infty))} = \mathbb{Z}(p^\infty)$. Considering a module $\mathbb{Z}$ being a closely simple module, then we have an empty set $Y = \{I_P | P \text{ is a sum-prime submodule of } \mathbb{Z}\} = \emptyset \subseteq \text{Spec}(\text{End}_{\mathbb{Z}}(\mathbb{Z}))$. Considering the socle $\text{soc}(M) \leq M$ as an $R$–submodule of any module $R M$, let $T$ denote the endomorphism ring $\text{End}_R(\text{soc}(M))$ of $R \text{soc}(M)$.

**Theorem 8.8.** For a closedly regular module $R M$ with the prime socle $\text{soc}(M)$, if $\{I_L | L \in \text{Spec}_N(\text{soc}(M))\}$ is open dense in Spec$(T)$, where $T = \text{End}_R(\text{soc}(M))$ is the endomorphism ring of the submodule $\text{soc}(M)$, the following are equivalent:

1. The endomorphism ring $\text{End}_R(\text{soc}(M))$ is prime;
2. The prime spectrum Spec$(T)$ is irreducible;
3. The kernel(null) spectrum Spec$_N(\text{soc}(M))$ is irreducible;
4. The prime socle $\text{soc}(M)$ of $R M$ is sum-prime.

**Proof.** (1) $\Rightarrow$ (2): It is trivial.
(2) $\Rightarrow$ (3): Assume (1), then the prime spectrum Spec$(T)$ is irreducible. Thus by the above Lemma 8.4 we have the irreducible kernel(null) spectrum Spec$_N(\text{soc}(M))$.
(3) $\Rightarrow$ (4): From Corollary 8.4 it follows immediately.
(4) $\Rightarrow$ (1): Assume that the kernel(null) spectrum Spec$_N(\text{soc}(M))$ is irreducible, then the prime socle $\text{soc}(\text{soc}(M)) = \text{soc}(M) \in \text{Spec}_N(\text{soc}(M))$ is a fully invariant sum-prime submodule of a closedly regular module $\text{soc}(M)$. Therefore we obtain a prime ideal $I_{\text{soc}(M)} = I_M = 0 \leq T$ of the endomorphism ring of $\text{soc}(M)$. Therefore the endomorphism ring $T$ is a prime ring. 

**Theorem 8.9.** For a closedly regular module $R M$ with the prime socle $\text{soc}(M)$, if $\{I_L | L \in \text{Spec}_N(\text{soc}(M))\}$ is open dense in Spec$(T)$, where $T = \text{End}_R(\text{soc}(M))$ is the endomorphism ring of the submodule $\text{soc}(M)$, the following are equivalent:

1. The endomorphism ring $\text{End}_R(\text{soc}(M))$ is not prime;
2. The prime spectrum Spec$(T)$ is reducible;
3. The kernel(null) spectrum Spec$_N(\text{soc}(M))$ is reducible;
4. The prime socle $\text{soc}(M)$ of $R M$ is not sum-prime.
Theorem 8.10. For a closedly regular module $R M$ with $\text{soc}(M) = M$, if 
\{I_P | P \in \text{Spec}_N(M)\} \text{ is open dense in Spec}(S)$, then the following are equivalent:

1. The endomorphism ring $S$ is prime;
2. The prime spectrum $\text{Spec}(S)$ is irreducible;
3. The kernel (null) spectrum $\text{Spec}_N(M)$ is irreducible;
4. $R M$ is sum-prime.

Proof. Replacing $\text{soc}(M)$ with $M$ in the above Theorem 8.7, the proof is completed. □

Theorem 8.11. For a closedly regular module $R M$ with $\text{soc}(M) = M$, if 
\{I_P | P \in \text{Spec}_N(M)\} \text{ is open dense in Spec}(S)$, then the following are equivalent:

1. The endomorphism ring $S$ is not prime;
2. The prime spectrum $\text{Spec}(S)$ is reducible;
3. The kernel (null) spectrum $\text{Spec}_N(M)$ is reducible;
4. $R M$ is not sum-prime.

9. Zariski topologies for commutators of rings

For a left $R$–module $R M$ over a ring $R$, let $Z$ denote the commutator of the ground ring $R$ over which $R M$ is a left $R$–module,

that is, $Z = \{a \in R | ar = ra, \text{ for each } r \in R\}$.

We are regarding any left multiplication by $a \in Z$, denoted by $\rho(a) : R M \to R M$ defined by $m \rho(a) = am$ for every element $m \in M$ as an endomorphism, in other words, $\rho(Z) = \{ \rho(a) | a \in Z \} \leq \text{End}_R(M)$ is a subring with identity of the endomorphism $\text{End}_R(M)$. Moreover for any left $R$–module $R M$ over a commutative ring $R$ with identity, clearly it follows that $Z = R$ and $\rho(R) = \{ \rho(r) | r \in R \} \leq \text{End}_R(M)$ is a subring of the endomorphism $\text{End}_R(M)$.

Thus if $P \leq R M$ is a meet-[resp. sum]-prime submodule of $R M$, we have a prime ideal $I_P \cap \rho(Z)$ [resp. $I_P \cap \rho(Z)] \leq \rho(Z)$ of the subring $\rho(Z)$ of the endomorphism $\text{End}_R(M)$, for all modules over any ring $R$ with identity.

It is well-known that any commutative ring $R$ can construct the Zariski topology of the prime spectrum $\text{Spec}(R) = \{ J \vartriangleleft R | J \text{ is a prime ideal of } R \}$, by the same method we can construct the Zariski topology of the prime spectrum $\text{Spec}(\rho(Z))$, if as closed sets we take all sets of form $v(E) = \{ I \in$
Spec(ρ(Z)) | E ⊆ I}, where E is any subset of ρ(Z). Precisely, the sets v(E) satisfy the axioms for closed sets in a topological space:

1. For any subset E ⊆ ρ(Z), if ⟨E⟩ is the ideal of ρ(Z) generated by E, then v(E) = v(⟨E⟩) = v(r(E)), where r(E) = \bigcap_{E \subseteq J_\alpha \in \text{Spec}(\rho(Z))} J_\alpha is the prime radical of E.
2. v(0) = Spec(ρ(Z)), v(ρ(Z)) = ∅.
3. v(\bigcup_{i \in I} E_i) = \bigcap_{i \in I} v(E_i), for each E_i ⊆ ρ(Z).
4. v(AB) = v(A) ∪ v(B) for A, B ⊆ ρ(Z).

**Theorem 9.1.** For any module \( R \mathcal{M} \) over a ring R with identity, the following are equivalent:

1. Spec(ρ(Z)) is irreducible;
2. The prime radical \( \text{rad}(\rho(Z)) = \bigcap_{J \in \text{Spec}(\rho(Z))} J \) is in Spec(\( \rho(Z) \)), that is, \( \text{rad}(\rho(Z)) \) is a prime ideal of \( \rho(Z) \).

In fact, it is true that the prime radical

\[
\text{rad}(\rho(Z)) = \bigcap_{J \in \text{Spec}(\rho(Z))} J = \text{rad}(S) \cap \rho(Z),
\]

where \( \text{rad}(\rho(Z)) \) is the prime radical of \( \rho(Z) \) and \( \text{rad}(S) = \bigcap_{J \in \text{Spec}(S)} J \) is the prime radical of the endomorphism ring \( S \) of \( R \mathcal{M} \). The following note is rewritten for a faithful module \( R \mathcal{M} \) over a commutative ring R in terms of ρ(Z) = \( \rho(Z) \).

**Note 9.2.** For (any faithful module \( R \mathcal{M} \) over) a commutative ring R with identity, the following are equivalent:

1. Spec(\( R \)) is irreducible;
2. The prime radical \( \text{rad}(R) = \bigcap_{J \in \text{Spec}(R)} J \) is in Spec(\( R \)), i.e., \( \text{rad}(R) \) is a prime ideal of \( R \).

Since \( R \mathcal{M} \) is faithful we can identify the subring \( \rho(Z) \) of \( S \) with the ground ring \( R \). Replace \( \rho(Z) \) by \( R \).

**10. On openly regular modules**

For any fully invariant meet-prime submodule \( P \leq M \) of a module \( R \mathcal{M} \), we have prime ideals \( I^P \leq S \) and \( I^P \cap \rho(Z) \leq \rho(Z) \).
For any module \( R \cdot M \), we have a surjective mapping from the image spectrum \( \text{Spec}_I(M) \) onto a subset \( \{ I^P \mid P \in \text{Spec}_I(M) \} \subseteq \text{Spec}(S) \) of the prime spectrum \( \text{Spec}(S) \) of the endomorphism ring \( S \) of \( R \cdot M \). Also we have a surjective mapping from the image spectrum \( \text{Spec}_I(M) \) onto \( \{ I^P \cap \rho(Z) \mid P \in \text{Spec}_I(M) \} \subseteq \text{Spec}(\rho(Z)) \) of the prime spectrum \( \text{Spec}(\rho(Z)) \) of the commutator ring \( \rho(Z) \) of a ring \( R \) with identity.

Let this subspace \( \{ I^P \mid P \in \text{Spec}_I(M) \} \) be inherited from the Zariski topology of the spectrum \( \text{Spec}(S) \) of the endomorphism ring. Then we have the next results. No proof will be given.

**Theorem 10.1.** For any openly regular module \( R \cdot M \) if

\[
\{ I^P \mid P \in \text{Spec}_I(M) \} \quad \text{and} \quad \{ I^P \cap \rho(Z) \mid P \in \text{Spec}_I(M) \}
\]

are open dense sets in the prime spectra \( \text{Spec}(S) \) and \( \text{Spec}(\rho(Z)) \), respectively, then the following are equivalent:

1. The prime spectrum \( \text{Spec}(S) \) is irreducible;
2. The image spectrum \( \text{Spec}_I(M) \) is irreducible;
3. The prime spectrum \( \text{Spec}(\rho(Z)) \) is irreducible.

Note here if the commutator \( \rho(Z) \) is not a prime ring, then immediately follows that neither \( S \) nor \( R \) is a prime ring. Thus we have the following corollary of the contraposition of Theorem 10.1 as follows:

**Corollary 10.2.** For any openly regular module \( R \cdot M \), if

\[
\{ I^P \mid P \in \text{Spec}_I(M) \} \quad \text{and} \quad \{ I^P \cap \rho(Z) \mid P \in \text{Spec}_I(M) \}
\]

are open dense sets in the prime spectra \( \text{Spec}(S) \) and \( \text{Spec}(\rho(Z)) \), respectively, then the following are equivalent:

1. The prime spectrum \( \text{Spec}(S) \) is reducible;
2. The image spectrum \( \text{Spec}_I(M) \) is reducible;
3. The prime spectrum \( \text{Spec}(\rho(Z)) \) is reducible.

**Remark 10.3.** The opennesses and density of \( \{ I^P \mid P \in \text{Spec}_I(M) \} \) and \( \{ I^P \cap \rho(Z) \mid P \in \text{Spec}_I(M) \} \) in the hypotheses of the Theorem 10.1 and Corollary 10.2 is essential.
Theorem 10.4. For any openly regular module $R^M$ with $\text{rad}(M) = 0$, if 
$\{ I^P \mid P \in \text{Spec}_I(M) \}$ and $\{ I^P \cap \rho(Z) \mid P \in \text{Spec}_I(M) \}$ are open dense sets in the prime spectra $\text{Spec}(S)$ and $\text{Spec}(\rho(Z))$, respectively, then the following are equivalent:

1. The commutator $\rho(Z)$ has a prime annihilator ideal $\text{Ann}_R(M) \cap \rho(Z)$;
2. The endomorphism ring $S$ is prime;
3. The prime spectrum $\text{Spec}(S)$ is irreducible;
4. The prime spectrum $\text{Spec}(\rho(Z))$ is irreducible;
5. The image spectrum $\text{Spec}_I(M)$ is irreducible;
6. $0 \leq M$ is meet-prime.

Theorem 10.5. For any openly regular module $R^M$ with $\text{rad}(M) = 0$, if 
$\{ I^P \mid P \in \text{Spec}_I(M) \}$ and $\{ I^P \cap \rho(Z) \mid P \in \text{Spec}_I(M) \}$ are open dense sets in the prime spectra $\text{Spec}(S)$ and $\text{Spec}(\rho(Z))$, respectively, then the following are equivalent:

1. The commutator $\rho(Z)$ has a nonprime ideal $\text{Ann}_R(M) \cap \rho(Z)$;
2. The endomorphism ring $S$ is not prime;
3. $0$ is not meet-prime;
4. The prime spectrum $\text{Spec}(S)$ is reducible;
5. The prime spectrum $\text{Spec}(\rho(Z))$ is reducible;
6. The image spectrum $\text{Spec}_I(M)$ is reducible.

For any faithful module $R^M$, the annihilator $\text{Ann}_R(M) = 0$ is trivial. Thus we have immediate consequences of Theorem 10.4 and Corollary 10.5 as follows.

Corollary 10.6. For any openly regular faithful module $R^M$ with $\text{rad}(M) = 0$, if 
$\{ I^P \mid P \in \text{Spec}_I(M) \}$ and $\{ I^P \cap \rho(Z) \mid P \in \text{Spec}_I(M) \}$ are open dense sets in the prime spectra $\text{Spec}(S)$ and $\text{Spec}(\rho(Z))$, respectively, then the following are equivalent:

1. The commutator $\rho(Z)$ is prime;
2. The endomorphism ring $S$ is prime;
3. The prime spectrum $\text{Spec}(S)$ is irreducible;
4. The prime spectrum $\text{Spec}(\rho(Z))$ is irreducible;
5. The image spectrum $\text{Spec}_I(M)$ is irreducible;
6. $0 \leq M$ is meet-prime.
Corollary 10.7. For any openly regular faithful module $R M$ with $\text{rad}(M) = 0$, if $\{ I^P \mid P \in \text{Spec}(M) \}$ and $\{ I^P \cap \rho(Z) \mid P \in \text{Spec}(M) \}$ are open dense sets in the prime spectra $\text{Spec}(S)$ and $\text{Spec}(\rho(Z))$, respectively, then the following are equivalent:

1. The commutator $\rho(Z)$ is not prime;
2. The endomorphism ring $S$ is not prime;
3. $0 \leq M$ is not meet-prime;
4. The prime spectrum $\text{Spec}(S)$ is reducible;
5. The prime spectrum $\text{Spec}(\rho(Z))$ is reducible;
6. The image spectrum $\text{Spec}_I(M)$ is reducible.

11. On closedly regular modules

For any module $R M$, we have a surjective mapping from the kernel(null) spectrum $\text{Spec}_N(M)$ onto a subset $\{ I_P \mid P \in \text{Spec}_N(M) \} \subseteq \text{Spec}(S)$ of the prime spectrum $\text{Spec}(S)$ of the endomorphism ring $S$ of $M$.

Also we have a surjective mapping from the kernel(null) spectrum $\text{Spec}_N(M)$ onto a subset $\{ I_P \cap \rho(Z) \mid P \in \text{Spec}_N(M) \} \subseteq \text{Spec}(\rho(Z))$ of the prime spectrum $\text{Spec}(\rho(Z))$ of the commutator of ring $R$.

Let this subspace $\{ I_P \mid P \in \text{Spec}_N(M) \}$ be inherited from the Zariski topology of the spectrum $\text{Spec}(S)$ of the endomorphism ring $S$. Then we have the next theorem.

Lemma 11.1. For any closedly regular module $R M$, if $\{ I_P \mid P \in \text{Spec}_N(M) \}$ and $\{ I_P \cap \rho(Z) \mid P \in \text{Spec}_N(M) \}$ are open dense sets in the prime spectra $\text{Spec}(S)$ and $\text{Spec}(\rho(Z))$, respectively, then the following are equivalent:

1. The prime spectrum $\text{Spec}(\rho(Z))$ is irreducible;
2. The prime spectrum $\text{Spec}(S)$ is irreducible;
3. The kernel(null) spectrum $\text{Spec}_N(M)$ is irreducible.

Corollary 11.2. For any openly regular module $R M$, if $\{ I_P \mid P \in \text{Spec}_N(M) \}$ and $\{ I_P \cap \rho(Z) \mid P \in \text{Spec}_N(M) \}$ are open dense sets in the prime spectra $\text{Spec}(S)$ and $\text{Spec}(\rho(Z))$, respectively, then the following are equivalent:

1. The prime spectrum $\text{Spec}(\rho(Z))$ is reducible;
2. The prime spectrum $\text{Spec}(S)$ is reducible;
3. The kernel(null) spectrum $\text{Spec}_N(M)$ is reducible.
Remark 11.3. The openness and density of \( \{ IP \mid P \in \text{Spec}_N(M) \} \) and \( \{ IP \cap \rho(Z) \mid P \in \text{Spec}_N(M) \} \) in the hypotheses of the Theorem 11.1 and Corollary 11.2 is essential.

Theorem 11.4. For any closedly regular module \( R \) with \( \text{soc}(M) = M \), if \( \{ IP \mid P \in \text{Spec}_N(M) \} \) and \( \{ IP \cap \rho(Z) \mid P \in \text{Spec}_N(M) \} \) are open dense sets in the prime spectra \( \text{Spec}(S) \) and \( \text{Spec}(\rho(Z)) \), respectively, then the following are equivalent:

1. The commutator \( \rho(Z) \) has a prime ideal \( \text{Ann}_R(M) \cap \rho(Z) \);
2. The endomorphism ring \( S \) is prime;
3. The prime spectrum \( \text{Spec}(S) \) is irreducible;
4. The prime spectrum \( \text{Spec}(\rho(Z)) \) is irreducible;
5. The kernel (null) spectrum \( \text{Spec}_N(M) \) is irreducible;
6. \( M \leq M \) is sum-prime.

Theorem 11.5. For any closedly regular module \( R \) with \( \text{soc}(M) = M \), if \( \{ IP \mid P \in \text{Spec}_N(M) \} \) and \( \{ IP \cap \rho(Z) \mid P \in \text{Spec}_N(M) \} \) are open dense sets in the prime spectra \( \text{Spec}(S) \) and \( \text{Spec}(\rho(Z)) \), respectively, then the following are equivalent:

1. The commutator \( \rho(Z) \) has a nonprime ideal \( \text{Ann}_R(M) \cap \rho(Z) \);
2. The endomorphism ring \( S \) is not prime;
3. The prime spectrum \( \text{Spec}(S) \) is reducible;
4. The prime spectrum \( \text{Spec}(\rho(Z)) \) is reducible;
5. The kernel (null) spectrum \( \text{Spec}_N(M) \) is reducible;
6. \( M \leq M \) is not sum-prime.

Theorem 11.6. For any closedly regular faithful module \( R \) with \( \text{soc}(M) = M \), if \( \{ IP \mid P \in \text{Spec}_N(M) \} \) and \( \{ IP \cap \rho(Z) \mid P \in \text{Spec}_N(M) \} \) are open dense sets in the prime spectra \( \text{Spec}(S) \) and \( \text{Spec}(\rho(Z)) \), respectively, then the following are equivalent:

1. The commutator \( \rho(Z) \) is prime;
2. The endomorphism ring \( S \) is prime;
3. The prime spectrum \( \text{Spec}(S) \) is irreducible;
4. The prime spectrum \( \text{Spec}(\rho(Z)) \) is irreducible;
5. The kernel (null) spectrum \( \text{Spec}_N(M) \) is irreducible;
6. \( M \leq M \) is sum-prime.
Theorem 11.7. For any closedly regular faithful module $R \mathcal{M}$ with $\text{soc}(M) = M$, if \( \{ I_P \mid P \in \text{Spec}_N(M) \} \) and \( \{ I_P \cap \rho(Z) \mid P \in \text{Spec}_N(M) \} \) are open dense sets in the prime spectra $\text{Spec}(S)$ and $\text{Spec}(\rho(Z))$, respectively, then the following are equivalent:

1. The commutator $\rho(Z)$ is not prime;
2. The endomorphism ring $S$ is not prime;
3. The prime spectrum $\text{Spec}(S)$ is reducible;
4. The kernel(null) spectrum $\text{Spec}_N(M)$ is reducible;
5. The prime spectrum $\text{Spec}(\rho(Z))$ is reducible;
6. $M \leq M$ is not sum-prime.

References


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