CONTINUITY OF JORDAN \*\-HOMOMORPHISMS OF BANACH \*\-ALGEBRAS

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1. Introduction

Let \( T : A \rightarrow B \) be a homomorphism between Banach algebras \( A \) and \( B \). Suppose that \( \overline{T(A)} \) is semi-simple. Is \( T \) necessarily continuous?

This is perhaps the most interesting open question remains in automatic continuity theory for Banach algebras. (To see [4], Open questions (16)). The continuity of homomorphisms between certain Banach algebras has been considered by several authors ([6], [9], [1], [4], [7], [3], [5], etc.). In this note we prove the following result: Let \( A \) be a complex Banach \*\-algebra with continuous involution and let \( B \) be an \( A^*\)-algebra. Let \( T : A \rightarrow B \) be an jordan \*\-homomorphism such that \( \overline{T(A)} = B \). Then \( T \) is continuous (Theorem 2).

From above theorem some others results of special interest and some well-known results follow. (Corollaries 3, 4, 5, 6 and 7). We close this note with some generalizations and some remarks (Theorems 8, 9, 10 and question).

Throughout this note we consider only complex algebras. Let \( A \) and \( B \) be complex algebras. A linear mapping \( T \) from \( A \) into \( B \) is called jordan homomorphism if \( T(x^2) = (Tx)^2 \) for all \( x \) in \( A \). A linear mapping \( T : A \rightarrow B \) is called spectrally-contractive mapping if \( \rho(Tx) \leq \rho(x) \) for all \( x \) in \( A \), where \( \rho(x) \) denotes spectral radius of element \( x \). Any homomorphism algebra is a spectrally-contractive mapping. If \( A \) and \( B \) are \*\-algebras, then a homomorphism \( T : A \rightarrow B \) is called \*\-homomorphism if \( (Th)^* = Th \) for all self-adjoint element \( h \) in \( A \). Recall that a Banach \*\-algebras is a complex Banach algebra with an involution \*. An \( A^*\)-algebra \( A \) is a Banach \*\-algebra having an auxiliary norm \( |\cdot| \) which satisfies \( B^*\)-condition \( |x^*x| = \)

Received May 29, 1992.
A Banach $*$-algebra whose norm is an algebra $B^*$-norm is called $B^*$-algebra. The $*$-semi-simple Banach $*$-algebras and the semi-simple hermitian Banach $*$-algebras are $A^*$-algebras. Also, $A^*$-algebras include $B^*$-algebras ($C^*$-algebras). Recall that a semi-prime algebra is an algebra without nilpotents two-sided ideals non-zero. The class of semi-prime algebras includes the class of semi-prime algebras and the class of prime algebras. For all concepts and basic facts about Banach algebras we refer to [2] and [8].

2. Continuity of Jordan $*$-homomorphism

First, we give the following result:

**Theorem 1.** If $T$ is an Jordan homomorphism from a Banach algebra $A$, with its range dense into a semi-prime Banach algebra $B$, then $T$ is a spectrally-contractive mapping.

**Proof.** Using the properties of the Jordan homomorphisms it is readily verified that $T([x, y]) = [Tx, Ty]$ for all $x$ and $y$ in $A$, and $T([x, y], z) = [Tx, Ty, Tz]$ for all $x$, $y$ and $z$ in $A$, where $[x, y] = xy - yx$ denotes Lie product of elements $x$ and $y$. Now, let $x$ and $y$ be in $A$ such that $[x, y] = 0$. Then it follows $[Tx, Ty]^2 = 0$. The continuity of Lie multiplication and density of range of $T$ imply that $[Tx, Ty]$ is a central nilpotent element of $B$. But a semi-prime algebra contains not of central nilpotent non-zero elements. Therefore $[x, y] = 0$ implies that $[Tx, Ty] = 0$. Hence, if $xy = yx$, then $T(xy) = T(yx) = \frac{1}{2}T(xy + yx) = \frac{1}{2}(TxTy + TyTx) = TxTy = TyTx$. If $x$ is an arbitrary quasi-invertible element of $A$, then there exists an element $y$ in $A$ such that $xy = yx = x + y$. It follows that $TxTy = TyTx = Tx + Ty$, that is $Ty$ is quasi-invertible element of $Tx$. Hence $T$ reduces the spectrum of elements. From this the conclusion of theorem it follows.

**Theorem 2.** If $T$ is an Jordan $*$-homomorphism from a Banach $*$-algebra $A$ with continuous involution, with its range dense into an $A^*$-algebra $B$, then $T$ is continuous.

**Proof.** $A^*$-algebras are semi-simple ([8], Theorem 4.1.19) and semi-simple algebras are semi-prime ([2], proposition 30.4). Then, by Theorem 1, $\rho(Tx) \leq$
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\(\rho(x)\) for all \(x\) in \(A\). Let \(h\) be a self-adjoint element of \(A\). Then, from [8], Lemma 4.1.14, it follows that \(|Th| \leq \rho(Th)\). Therefore, we have \(|Th| \leq \rho(Th) \leq \rho(h) \leq ||h||\) for each self-adjoint element \(h\) in \(A\). Every element \(x\) of a complex \(\ast\)-algebra \(A\) has an unique representation \(x = \mu + iv\), with \(\mu\) and \(v\) self-adjoint elements of \(A\). Since the involution is continuous, it follows that any sequence \((x_n)\) of elements of \(A\) converges to an element \(x\) in \(A\) if only if the sequence of self-adjoint components of \((x_n)\) converge respectively to corresponding self-adjoint components of \(x\). Let \((x_n)\) be a sequence of elements in \(A\) such that \(x_n \to 0\) as \(n \to \infty\). If \(x_n = \mu_n + v_n\), where \(\mu_n^* = \mu_n\) and \(v_n^* = v_n\), \(n = 1, 2, 3, \ldots\), then \(|Tx_n| \leq |T\mu_n| + |Tv_n| \leq ||\mu_n|| + ||v_n|| \to 0\), as \(n \to \infty\). Thus, by Closed Graph Theorem, it follows that \(T\) is continuous.

**Corollary 3.** If \(T\) is an Jordan \(\ast\)-homomorphism from a semi-simple Banach \(\ast\)-algebra, with its range dense into an \(A^*\)-algebra, then \(T\) is continuous.

**Proof.** On a semi-simple Banach algebra every involution is continuous ([6]). Apply Theorem 2.

**Corollary 4.** If \(T\) is an Jordan \(\ast\)-homomorphism between \(A^*\)-algebras (in particular, \(B^*\)-algebras, \(C^*\)-algebras), with its range dense, then \(T\) is continuous.

**Proof.** The involution in an \(A^*\)-algebra is necessarily continuous with respect to both norms ([8], Theorem 4.1.15). Apply Theorem 2.

All the above results remain valid for any \(\ast\)-homomorphism \(T\), without density assumption on the range of \(T\).

**Corollary 5.** Any \(\ast\)-homomorphism from a Banach \(\ast\)-algebra with continuous involution into an \(A^*\)-algebra is continuous.

**Proof.** It is well-known that the homomorphisms reduce the spectra of elements.

**Corollary 6.** Any \(\ast\)-homomorphism from a semi-simple Banach \(\ast\)-algebra into an \(A^*\)-algebra is continuous.
Proof. By proof of Corollary 3.

Corollary 7. Any $^*$-homomorphism between $A^*$-algebras (in particular, $B^*$-algebras, $C^*$-algebras) is continuous.

Proof. It is clear.

3. Some generalizations and remarks.

We close this note with some generalizations and remarks which are of some interest. Theorem below is a generalization of Theorem 2 and it has a more elementary proof.

Theorem 8. Let $A$ be a Banach $^*$-algebra with continuous involution, let $B$ be an $A^*$-algebra and let $T : A \rightarrow B$ be a linear mapping such that $T h$ is a self-adjoint element of $B$ and $\rho(T h) \leq \rho(h)$ for every self-adjoint element $h$ of $A$. Then $T$ is continuous.

We do not know if this theorem holds without continuity assumption on the involution in $A$, but have the following partial result which extends Corollary 5:


Actually it is desirable to consider the more general situation.

Theorem 10. Let $A$ be a Banach algebra, and let $T$ be a homomorphism from $A$ into an $A^*$-algebra.

Assume that for each element $x$ in $A$ there exists an element $y$ in $A$ such that $\|yx\| \leq \|x\|^2$ and $(Tx)^* = Ty$. Then $T$ is continuous.

The condition $\|yx\| \leq \|x\|^2$ is practically redundant here and removing it we obtain a generalization of the Theorem 3.2 of [7]. It only remains to give an elementary proof.

Finally, we think that is special interest to know the answer to

Question: Is every spectrally-contracive linear mapping from a Banach algebra into an $A^*$-algebra necessarily continuous?
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References


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