A REMARK ON MULTIPLICATION MODULES

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Modules which satisfy the converse of Schur's lemma have been studied by many authors. In [6], R. Ware proved that a projective module \( P \) over a semiprime ring \( R \) is irreducible if and only if \( \text{End}_R(P) \) is a division ring. Also, Y. Hirano and J.K. Park proved that a torsionless module \( M \) over a semiprime ring \( R \) is irreducible if and only if \( \text{End}_R(M) \) is a division ring. In case \( R \) is a commutative ring, we obtain the following: An \( R \)-module \( M \) is irreducible if and only if \( \text{End}_R(M) \) is a division ring and \( M \) is a multiplication \( R \)-module.

Throughout this paper, \( R \) is a commutative ring with identity and all modules are unital left \( R \)-modules.

Let \( R \) be a commutative ring with identity and let \( M \) be an \( R \)-module. Then \( M \) is called a multiplication module if for each submodule \( N \) of \( M \), there exists an ideal \( I \) of \( R \) such that \( N = IM \).

Cyclic \( R \)-modules are multiplication modules. In particular, irreducible \( R \)-modules are multiplication modules.

An endomorphism \( f \) of an \( R \)-module \( M \) is called trivial if there exists \( a \in R \) such that \( f(m) = am \), for all \( m \in M \). We set

\[
\text{Tri}(M) = \{ f \in \text{End}_R(M) | f \text{ is trivial} \}.
\]

Clearly, \( \text{Tri}(M) \cong R/\text{ann}_R(M) \) and \( \text{Tri}(M) \) is a subring of \( \text{End}_R(M) \).

**Theorem 1.** Let \( M \) be a finitely generated faithful multiplication \( R \)-module. If \( \text{End}_R(M) \) is a division ring, then \( R \) is a field and \( M \) is an \( 1 \)-dimensional vector space over \( R \).
Proof. Let $M$ be a finitely generated faithful multiplication $R$-module. Then $R \cong \text{Tri}(M)$, and $R$ can be embedded in $\text{End}_R(M)$. Since $M$ is finitely generated multiplication $R$-module, $\text{End}_R(M) = \text{Tri}(M)$ [5, Theorem 3]. Hence $R \cong \text{End}_R(M)$. Thus $R$ is a field. In view of Corollary in [3], $M$ is an 1-dimensional vector space over $R$.

**Lemma 2.** Let $M$ be a faithful $R$-module. Then $M$ is irreducible if and only if $\text{End}_R(M)$ is a division ring and $M$ is a multiplication $R$-module.

**Proof.** It remains to show the “if” part. Assume that $M$ is a faithful multiplication $R$-module and let $\text{End}_R(M)$ be a division ring. Since $M$ is a faithful $R$-module, $R$ can be embedded in $\text{End}_R(M)$. Since $\text{End}_R(M)$ is a division ring, $R$ is an integral domain. Thus $R$ is a semiprime ring. Now $M$ is a faithful multiplication $R$-module, $M$ is torsionless [4, Theorem 1.3]. By [2, Corollary 3], $M$ is irreducible.

**Theorem 3.** Let $M$ be an $R$-module. Then $M$ is irreducible if and only if $\text{End}_R(M)$ is a division ring and $M$ is a multiplication $R$-module.

**Proof.** Let $I = \text{ann}_R(M)$. Then $M$ is a faithful $R/I$-module and $M$ is a multiplication $R/I$-module. Since $\text{End}_{R/I}(M) = \text{End}_R(M)$ and $\text{End}_R(M)$ is a division ring, $\text{End}_{R/I}(M)$ is a division ring. By Lemma 2, $M$ is a irreducible $R/I$-module. Thus $M$ is a irreducible $R$-module.

**Corollary.** Let $M$ be a projective $R$-module. Then $M$ is irreducible if and only if $\text{End}_R(M)$ is a divison ring.

**Proof.** If $M$ is a projective $R$-module and $\text{End}_R(M)$ is a division ring, then $M$ is a cyclic module [6, Proposition 4.3]. Thus $M$ is a multiplication $R$-module. By Theorem 3, $M$ is irreducible.

**References**

A remark on multiplication modules


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