A CHARACTERIZATION OF PROJECTIVE GEOMETRIES

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ABSTRACT. It is known that a projective geometry \( PG(n - 1, q) \) is upper homogeneous, has a modular copoint, and its characteristic polynomial is \((\lambda - 1)(\lambda - q)(\lambda - q^2) \ldots (\lambda - q^{n-1}) \). We prove the converse of the above statement.

1. Introduction

The most fundamental examples of (combinatorial) geometries are projective geometries \( PG(n - 1, q) \) of dimension \( n - 1 \), representable over \( GF(q) \), where \( q \) is a prime power. Every upper interval of a projective geometry is a projective geometry. The Whitney numbers of the second kind are the Gaussian coefficients. Every flat of a projective geometry is modular, so the projective geometry is supersolvable in the sense of Stanley [6].

The characteristic polynomial \( p(G, \lambda) \) of a geometry \( G \) of rank \( n \) is defined by

\[
p(G, \lambda) = \sum_{a \in L(G)} \mu(\hat{0}, a)\lambda^{n-r(a)}
\]

where \( L(G) \) is the lattice of flats of \( G \) and \( \mu \) is the Möbius function of \( L(G) \).

In this paper, we give a characterization of projective geometries in terms of their characteristic polynomials and some other conditions.

Our notation and terminology follow those in [7,8]. To clarify our terminology, let \( G \) be a finite geometric lattice. If \( S \) is the set of points (or rank-one flats) in \( G \), the lattice structure of \( G \) induces the structure of a (combinatorial) geometry, also denoted by \( G \), on \( S \). The size \( |G| \) of the geometry \( G \) is the number of points in \( G \). Let \( T \) be a subset of \( S \). The deletion of \( T \) from \( G \) is the geometry on the point set \( S \setminus T \) obtained by restricting \( G \) to the subset \( S \setminus T \). The contraction \( G/T \) of \( G \) by \( T \) is the geometry induced by the geometric
lattice \([cl(T), \hat{1}]\) on the set \(S'\) of all flats in \(G\) covering \(cl(T)\). (Here, \(cl(T)\) is the closure of \(T\), and \(\hat{1}\) is the maximum of the lattice \(G\).) Thus, by definition, the contraction of a geometry is always a geometry. A geometry which can be obtained from \(G\) by deletions or contractions is called a \textit{minor} of \(G\).

2. Preliminaries

A geometry \(G\) is said to be \textit{upper homogeneous} if for \(k = 1, 2, \ldots, r(G)\), \(G/x \cong G/y\) for every pairs \(x, y\) of flats of rank \(k\). Kahn and Kung [4] defined splitting in geometries. A geometry \(G\) splits if \(G\) is the union of two of its proper flats. And \(G\) is said to be \textit{non-splitting} otherwise.

\textsc{Lemma 2.1.} [9] If a geometry \(G\) is upper homogeneous, has a modular copoint, and \(|G| > r(G)\), then \(G\) is non-splitting.

\textsc{Lemma 2.2.} Let \(G\) be an upper homogeneous geometry having a modular copoint. Then \(G\) is supersolvable. Let \(\emptyset < x_1 < x_2 < \cdots < x_{n-1} < x_n = G\) be a maximal chain of modular flats of \(G\). Let \(a_i\) be the number of points in \(x_i\) but not in \(x_{i-1}\) for each \(i = 2, 3, \ldots, n\). Then we have \(a_i \leq a_{i+1}\) for each \(i = 1, 2, \ldots, n - 1\).

\textit{Proof.} Let \(n\) be the rank of \(G\) and let \(x_{n-1}\) be a modular copoint of \(G\). Then \([\hat{0}, x_{n-1}] \cong G/a\) for a point \(a\) not in \(x_{n-1}\). Since \(G\) is upper homogeneous, it follows that \([\hat{0}, x_{n-1}] \cong G/b\) for a point \(b\) in \(x_{n-1}\). Thus \(x_{n-1}\) is upper homogeneous and has a modular copoint \(x_{n-2}\) of \(x_{n-1}\) such that \([\hat{0}, x_{n-2}] \cong x_{n-2}/b\). It follows that \(x_{n-2}\) is a modular coline of \(G\). By repeating the same arguments, we have a maximal chain \(\emptyset < x_1 < x_2 < \cdots < x_{n-1} < G\) of modular flats in \(G\). Thus \(G\) is supersolvable. Let \(a\) be a point in \(x_i\) but not in \(x_{i-1}\) for some \(i\). Since \(x_{i+1}/a \cong [\hat{0}, x_i]\) and \(x_i/a \cong [\hat{0}, x_{i-1}]\), it implies that \(a_i = |x_i| - |x_{i-1}| \leq |x_{i+1}| - |x_i/a| = |x_{i+1}| - |x_i| = a_{i+1}\). Thus \(a_i \leq a_{i+1}\) for each \(i = 1, 2, \ldots, n - 1\).

A geometry is \textit{modular} if all of its flats are modular. The following propositions give characterizations of modular geometries.

\textsc{Proposition 2.3.} [1] A geometry is modular if and only if it is the direct sum of projective geometries or points.
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PROPOSITION 2.4. [3] A geometry $G$ is modular if and only if the number of points in $G$ is the same as the number of copoints in $G$.

The Whitney numbers of a geometry $G$ of rank $n$ are defined by

$$w(n, s) = \sum_{r(x)=n-s} \mu(\hat{0}, x),$$

the coefficient of $\lambda^s$ in the characteristic polynomial; and

$$W(n, s) = \sum_{r(x)=n-s} 1,$$

the number of flats of rank $n - s$. The most well-known examples are the following (See Dowling[2, p.75]):

1. If $G = B_n$, the Boolean algebra of rank $n$, then

$$w(n, s) = (-1)^{n-s} \binom{n}{s} \quad \text{and} \quad W(n, s) = \binom{n}{s}.$$ 

2. If $G = PG(n-1, q)$, then

$$w(n, s) = (-1)^{n-s} q^{\binom{n-s}{q}} \binom{n}{s} \quad \text{and} \quad W(n, s) = \binom{n}{s}_q,$$

where $\binom{n}{s}_q$ is the Gaussian coefficient,

$$\binom{n}{s}_q = \frac{(q^n - 1) \ldots (q^{n-s+1} - 1)}{(q^s - 1) \ldots (q - 1)}.$$

Each of these examples are classes of geometries which satisfy the hypotheses of the following theorem due to Dowling.

THEOREM 2.5. [2] Let $\{G_n : n = 1, 2, \ldots\}$ be a class of geometries such that $G_n$ is of rank $n$, and, for all flats $x$ in $G_n$ of rank $n - s$ $(0 \leq s \leq n)$,
the interval \([x, \hat{1}]\) is isomorphic to \(G_s\). Let \(w(n, s)\), \(W(n, s)\) be the Whitney numbers of \(G_n\). Then

\[
\sum_s W(n, s)w(s, t) = \delta(n, t),
\]

\[
\sum_s w(n, s)W(s, t) = \delta(n, t),
\]

and the numbers \(w(n, s), W(n, s)\) satisfy the inverse relations

\[
a_n = \sum_s W(n, s)b_s, \quad b_n = \sum_s w(n, s)a_s.
\]

3. Main Theorem

THEOREM 3.1. Let \(q\) be a power of prime. If a geometry \(G\) is upper homogeneous, has a modular copoint, and \(p(G; \lambda) = (\lambda - 1)(\lambda - q)(\lambda - q^2)\ldots(\lambda - q^{n-1})\), then \(G \cong PG(n - 1, q)\).

Proof. By Lemma 2.2, \(G\) is supersolvable. Let \(\emptyset < x_1 < x_2 < \ldots < x_{n-1} < G\) be a maximal chain of modular flats of \(G\). Let \(a_i\) be the number of points in \(x_i\) but not in \(x_{i-1}\) for \(i = 2, 3, \ldots, n\). Then the modular factorization theorem [5] implies that \(p(G; \lambda) = (\lambda - 1)(\lambda - a_2)(\lambda - a_3)\ldots(\lambda - a_n)\). By Lemma 2.2, we have \(a_i \leq a_{i+1}\) for each \(i = 1, 2, \ldots, n - 1\). Thus we can conclude that \(a_i = q^{i-1}\) for \(i = 2, 3, \ldots, n\).

We prove this theorem by induction on \(n\). For \(n = 1\) and \(n = 2\), the theorem is true. Assume it holds for a geometry of rank less than \(n\). Let \(a\) be a point in \(G\). Then \(G/a\) is upper homogeneous and has a modular copoint and \(p(G/a; \lambda) = p(x_{n-1}; \lambda) = (\lambda - 1)(\lambda - q)(\lambda - q^2)\ldots(\lambda - q^{n-2})\). By the induction hypothesis, \(G/a \cong PG(n - 2, q)\) for every point \(a\) in \(G\).

Since projective geometries are modular, Proposition 2.4 implies that \(W(s, 1)\) is the same as the number of points in \(PG(s - 1, q)\). Thus \(W(s, 1) = \frac{q^s - 1}{q - 1} = \binom{s}{1}_q\) for \(s = 1, 2, \ldots, n - 1\). By Theorem 2.5, we have

\[
\sum_s w(n, s)W(s, t) = \delta(n, t).
\]
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Let \( t = 1 \) and \( n > 1 \). Then we have

\[
W(n, 1) = - \sum_{s=1}^{n-1} w(n, s) W(s, 1)
\]

\[
= - \sum_{s=0}^{n-1} (-1)^{n-s} q^{(n-s)/2} \binom{n}{s} \binom{s}{1}_q
\]

\[
= - \sum_{s=0}^{n} (-1)^{n-s} q^{(n-s)/2} \binom{n}{s} \binom{s}{1}_q + \binom{n}{1}_q = \binom{n}{1}_q
\]

\[
= W(n, n - 1).
\]

Thus Proposition 2.4 implies that \( G \) is modular. Also Lemma 2.1 implies that \( G \) is non-splitting and so \( G \) is connected. Since \( G \) is a connected modular geometry, by Proposition 2.3, we can conclude that \( G \) is isomorphic to a projective geometry. Therefore \( G \cong PG(n - 1, q) \).

References


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