THE RANDERS CHANGES OF FINSLER SPACES
WITH \((\alpha, \beta)\)-METRICS OF DOUGLAS TYPE

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Abstract. A change of Finsler metric \(L(x,y) \rightarrow \mathcal{T}(x,y)\) is called a Randers change of \(L\), if \(\mathcal{T}(x,y) = L(x,y) + \rho(x,y)\), where \(\rho(x,y) = \rho_i(x)y^i\) is a 1-form on a smooth manifold \(M^n\). Let us consider the special Randers change of Finsler metric \(L \rightarrow \mathcal{T} = L + \beta\) by \(\beta\). On the basis of this special Randers change, the purpose of the present paper is devoted to studying the conditions for Finsler space \(F^n\) which are transformed by a special Randers change of Finsler spaces \(F^n\) with \((\alpha, \beta)\)-metrics of Douglas type to be also of Douglas type, and vice versa.

1. Introduction

An \(n\)-dimensional Finsler space \(F^n\) is a Douglas space or of Douglas type if and only if the Douglas tensor vanishes identically. Recently R. Bácsó and M. Matsumoto ([2]) have introduced the notion of Douglas space as a generalization of Berwald space from the viewpoint of geodesic equations. The conditions for some Finsler spaces with an \((\alpha, \beta)\)-metric to be Douglas space are obtained by M. Matsumoto ([8]).

A change of Finsler metric \(L(x,y) \rightarrow \mathcal{T}\) is called a Randers change of \(L\), if \(\mathcal{T}(x,y) = L(x,y) + \rho(x,y)\), where \(\rho(x,y) = \rho_i(x)y^i\) is a 1-form on a smooth manifold \(M^n\). The notion of a Randers change has been proposed by M. Matsumoto ([5]). If \(L(x,y)\) is a Riemannian metric, then \(\mathcal{T}(x,y)\) becomes the Randers metric.

The purpose of the present paper is to study the Randers change of the Finsler space which is Douglas type. After the section 4, we consider

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a special Randers change of certain Finsler spaces with an \((\alpha, \beta)\)-metric \(L\) by \(\beta\). The 1-form \(\beta\) of modification is coincided with 1-form \(\beta\) of \((\alpha, \beta)\)-metric \(L\). We are devoted to finding the conditions for Finsler spaces changed by a special Randers change to be of Douglas type.

2. Preliminaries

The geodesics of an \(n\)-dimensional Finsler space \(F^n = (M^n, L)\) are given by the system of the differential equations ([1]):

\[
\frac{d^2 x^i}{dt^2} y^j - \frac{d^2 x^j}{dt^2} y^i + 2 \{G^i(x,y) y^j - G^j(x,y) y^i\} = 0, \quad y^i = \frac{dx^i}{dt}
\]

in a parameter \(t\). The function \(G^i(x,y)\) are given by

\[
2G^i(x,y) = g^{ij} (\dot{y}^j \partial_j F - \partial_j F),
\]

where \(\dot{\partial}_l = \partial/\partial y^i, \partial_i = \partial/\partial x^i, F = L^2/2\) and \(g^{ij}(x,y)\) are the inverse of Finsler metric tensor \(g_{ij}(x,y)\). According to [2], \(F^n\) is of Douglas type if

\[
D^{ij} = G^i(x,y) y^j - G^j(x,y) y^i
\]

are homogeneous polynomials in \((y^i)\) of degree three. We shall denote the homogeneous polynomials in \((y^i)\) of degree \(r\) by \(h^r\) for brevity.

Let \(L_i = \dot{\partial}_i L, \quad L_{ij} = \dot{\partial}_i \dot{\partial}_j L, \quad L_{ijk} = \dot{\partial}_k \dot{\partial}_j \dot{\partial}_i L\). Then we have

\[
L_i = l_i, \quad LL_{ij} = h_{ij}, \quad L^2 L_{ijk} = h_{ij}l_k + h_{jk}l_i + h_{ki}l_j.
\]

And we put

\[
2E_{ij} = \rho_{ij} + \rho_{ji}, \quad 2F_{ij} = \rho_{ij} - \rho_{ji},
\]

where \((\cdot)\) denotes the \(h\)-covariant derivative with respect to the Cartan connection \(C^r = (F_{k^r}, G^r, C_{k^r}^r)\).

On the other hand, a Finsler metric \(L(x,y)\) is called an \((\alpha, \beta)\)-metric, when \(L\) is a positively homogeneous function \(L(\alpha, \beta)\) of degree one in two variables \(\alpha(x,y) = \sqrt{a_{ij}(x)y^i y^j}\) and \(\beta(x,y) = b_i(x)y^i\). The space \(R^n = (M^n, \alpha)\) is called the associated Riemannian space with \(F^n\) ([1], [7]).
We have the covariant differentiation (\(\cdot,\)) with respect to the Christoffel symbols \(\gamma^{ij}_{k}(x)\) in \(\mathbb{R}^n\). We shall use the symbols as follows:

\[
\begin{align*}
    r_{ij} &= \frac{1}{2}(b_{i,j} + b_{j,i}), \\
    s_{ij} &= \frac{1}{2}(b_{i,j} - b_{j,i}), \\
    s^i_j &= a^r s_{rj}, \\
    s_j &= b_{r}s^r_j.
\end{align*}
\]

Now we consider the functions \(G^i(x, y)\) of \(F^n\) with an \((\alpha, \beta)\)-metric. According to [8], \(G^i(x, y)\) are written in the form

\[
(2.3) \quad 2G^i = \gamma^i_{\cdot 0} + 2B^i,
\]

where \(L_\alpha = \partial L/\partial \alpha, L_\beta = \partial L/\partial \beta, L_{\alpha\alpha} = \partial^2 L/\partial \alpha \partial \alpha\), the subscript 0 means contraction by \(y^i\) and

\[
C^* = \frac{\alpha \beta (r_{00} L_\alpha - 2 \alpha s_0 L_\beta)}{2 (\beta^2 L_\alpha + \alpha \gamma^2 L_{\alpha\alpha})}, \quad \gamma^2 = b^2 \alpha^2 - \beta^2,
\]

\[
b^i = a^{ij} b_j, \quad b^2 = a^{ij} b_i b_j.
\]

Since \(\gamma^i_{\cdot 0}(x)\) are \(hp(2), F^n\) with an \((\alpha, \beta)\)-metric is Douglas space, if and only if \(B^i = B^i y^j - B^j y^i\) are \(hp(3)\). Form (2.1) and (2.3) we have

\[
(2.4) \quad B^i = \frac{\alpha L_\beta}{L_\alpha} (s^i_0 y^j - s^j_0 y^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha} C^* (b^i y^j - b^j y^i).
\]

The following lemma ([9]) is used for latter:

**Lemma.** A system of linear equations \(L_{ir} X^r = Y_i, (i_r + \rho_r) X^r = Y \) and \((Y_i y^i = \alpha^2)\) in \(X^i\) has the unique solution \(X^i = L Y^i + \frac{1}{2} (Y - LY^r \rho_r) l^i\), where \(Y^i = g^{ir} Y_r \) and \(\tau = L/L\).

### 3. Randers change of Douglas type

For a Randers change: \(L \rightarrow \overline{L} = L(x, y) + \rho(x, y), \rho(x, y) = \rho(x) y^i\), we may put

\[
(3.1) \quad \overline{G}^i = G^i + D^i.
\]
Then $\mathbf{G}_{ij} = G_{ij} + D_{ij}$ and $\mathbf{G}_{ijk} = G_{ijk} + D_{ijk}$, where $D_{ij} = \dot{\partial}_j D^i$ and $D_{ijk} = \dot{\partial}_k D_{ij}$. The tensors $D^i$, $D_{ij}$ and $D_{ijk}$ are positively homogeneous in $y^i$ of degree two, one and zero respectively. In the following the explicit form of $D^i$ is necessary. To find this, we deal with equation $L_{ijk} = 0$, where $L_{ijk}$ is the $h$-covariant derivative of $L_{ij} = h_{ij}/L$ in CT. Then

$$\partial_k L_{ij} = L_{i,jr} G^r_k + L_{rj} F^r_i + L_{ir} F^r_j.$$  

Since $L_{ij} = L_{ij}$ and $L_{ijk} = L_{ijk}$ hold,

$$L_{ijk} = L_{i,jr} (G^r_k + D^r_k) + L_{rj} (F^r_i - D^r_i) + L_{ir} (F^r_j + D^r_j),$$

which imply

$$L_{i,jr} D^r_k + L_{rj} D^r_i + L_{ir} D^r_j = 0.$$ 

Thus transvection of this equation by $y^k$ yields

$$2L_{i,jr} D^r + L_{rj} D^r_i + L_{ir} D^r_j = 0. \quad (3.2)$$

Next, we deal with $L_{ij} = 0$, that is,

$$\partial_j L_i = L_{ir} G^r_j + L_{rj} F^r_i,$$

$$\partial_j \mathcal{L}_i = L_{ir} (G^r_j + D^r_j) + (L_r + \partial_r c) (F^r_i + \epsilon D^r_i),$$

where $\epsilon D^r_i = \mathcal{F}_{ijr} - F^r_i$. Substitution of the equations above in $\partial_j \mathcal{L}_i = \partial_j L_i + \partial_j \rho_i$ leads to

$$\partial_j \rho_i - \rho_r F^r_i = L_{ir} D^r_j + (L_r + \partial_r c) \epsilon D^r_i.$$ 

Then we have

$$2E_{ij} = L_{ir} D^r_j + L_{jr} D^r_i + 2(l_r + \partial_r c) \epsilon D^r_j, \quad (3.3)$$

$$2F_{ij} = L_{ir} D^r_j - L_{jr} D^r_i. \quad (3.4)$$

Therefore (3.2) and (3.4) give

$$L_{ir} D^r_j = F_{ij} - L_{ijr} D^r_i \quad (3.5)$$

and transvection of (3.3) by $y^i$ shows

$$(l_r + \partial_r c) D^r_j = E_{ij} y^i - L_{jr} D^r. \quad (3.6)$$
Furthermore transvection of (3.5) and (3.6) by $y^i$ leads to

\[(3.7) \quad \begin{align*}
(a) \quad & L_{ir} D^r = F_{ij} y^j, \\
(b) \quad & (l_r + \rho_r) D^r = \frac{1}{2} E_{ij} y^j y^j.
\end{align*}\]

The equations (3.7)(a)(b) constitute a system of linear equations respectively. Applying Lemma to (3.7), we have

\[(3.8) \quad D^i = LF^i_0 + \frac{1}{L} \left( \frac{1}{2} E_{00} - LF_0 \right) y^i,
\]

where $F^i_j = g^{ir} F_{rj}$ and $F^i_j = \rho_r F^r_j$. Thus we have the following

**Proposition 3.1. ([9])** The tensor $D^i$ of (3.1) arising from a Randers change are given by (3.8).

From (3.1) and (3.8) we have

\[
\mathcal{G}^i_j y^j - \mathcal{G}^j_i y^i = G^i_j y^j - G^j_i y^i + L(F^i_0 y^j - F^j_0 y^i).
\]

Suppose $F^n$ is a Douglas space, that is, $G^i_j y^j - G^j_i y^i$ are $hp (3)$. Thus we have

**Proposition 3.2.** Let $F^n$ be a Douglas space and $\overline{F}^n$ a Finsler space which is obtained by Randers change by $\rho$. $\overline{F}^n$ is also a Douglas space if and only if $L(F^i_0 y^j - F^j_0 y^i)$ are $hp (3)$.

The Randers changes is called *projective Randers changes* if all the geodesic curves are preserved under the Randers changes. According to Hashiguchi-Ichijyo ([4]), a Randers change is projective, if and only if $p_i$ are gradient vector fields. In this case (3.8) is reduced to

\[
D^i = E_{00} y^i / 2L.
\]

Therefore $D^i y^i - D^i y^i = 0$. Thus we have

\[
\overline{\mathcal{G}}^i_j y^j - \mathcal{G}^i_j y^j = G^i_j y^j - G^j_i y^i.
\]

On the other hand, it is well-known that the Douglas tensor is projectively invariant. Hence, if a Finsler space is projectively related to a Douglas space, then it is also a Douglas space. Thus, from Hashiguchi-Ichijyo’s theorem, we have the following

**Theorem 3.3.** Let $F^n(M^n, L) \rightarrow \overline{F}^n(M^n, L + \rho)$ be a projective Randers change. If $F^n$ is a Douglas space, then $\overline{F}^n$ is also a Douglas space, and vice versa.
4. Generalized Kropina spaces

Hereafter we consider a special Randers change of certain \((\alpha, \beta)\) metric as follows: \(L(\alpha, \beta) \rightarrow L = L(\alpha, \beta) + \beta\), that is, the 1-form \(\beta\) of modification coincides with 1-form \(\beta\) of \((\alpha, \beta)\)-metric. In this section we deal with a Finsler space \(F^n\) \((n > 2)\) with a generalized Kropina metric. The metric of \(F^n\) is \(L = \alpha^{1+m}\beta^{-m}\), where \(m\) is a constant \(\neq 0, -1\).

We consider the condition for a Finsler space \(F^n = (M^n, L + \beta)\) which is obtained by a special Randers change of a generalized Kropina space \(F^n = (M^n, L = \alpha^{1+m}\beta^{-m})\) to be of Douglas type. It has been known ([8]) that a generalized Kropina space is of Douglas space, where \(\alpha^2 \not\equiv 0 \pmod{\beta}\), if and only if \(b_{ij}\) are given by

\[(4.1) \quad s_{ij} = \frac{1}{b^2} (b^i s_j - b_j s_i),\]

\[(4.2) \quad r_{ij} = \frac{k}{m(1 + m)} \{ (1 - m) b_i b_j + mb^2 a_{ij} \} + \frac{1 - m}{(1 + m)b^2} (s_i b_j - s_j b_i).\]

For \(F^n\), (2.3) gives

\[(4.3) \quad 2\{(1 - m)b^2 + mb^2 \alpha^2\} \{(1 + m)b^2 \Sigma^j i\} + (ma^2 - \alpha^{1-m} \beta^{m+1}) (s^i_0 y^j - s^j_0 y^i) - ma^2 \{ (1 + m)r_{00} \beta \}
+2s_0 (ma^2 - \alpha^{1-m} \beta^{m+1}) (b^i y^j - b^j y^i) = 0,\]

which are equivalent to

\[(4.4) \quad 2\{(1 - m)b^2 + mb^2 \alpha^2\} \{(1 + m)b^2 \Sigma^j i\} + ma^2 (s^i_0 y^j - s^j_0 y^i) - ma^2 \{ (1 + m)r_{00} \beta + 2ms_0 \alpha^2 \} (b^i y^j - b^j y^i)
-2\alpha^{1-m} \beta^{m+1} \{ (1 - m)b^2 + mb^2 \alpha^2 \} (s^i_0 y^j - s^j_0 y^i)
-ms_0 \alpha^2 (b^i y^j - b^j y^i) = 0.\]

Then it will be better to divide our consideration into two cases as follows:

(I) \(\alpha^{1-m} \beta^{m+1}\) : rational in \((y')\), that is, \(m\) : odd integer,
(II) \(\alpha^{1-m} \beta^{m+1}\) : irrational in \((y')\), that is, \(m\) : the others.
The case (I) : First we are concerned with \( m \leq 1 \), where \( m \) is an odd integer. Multiplication of (4.1) by \( \beta^{-m-1} \) leads to

\[
2\{(1 - m^2)\beta^2 + mb^2\alpha^2\}(1 + m)\beta^{-m}\overline{B}^{ij} + (ma^2\beta^{-1-m} - \alpha^{-1-m})(s'_0y^j - s'_0y^i) - \alpha^{-1-m}(s'_0y^j - s'_0y^i) - ma^2{(1 + m)r_{00}\beta^{-m}} + 2s_0(ma^2\beta^{-1-m} - \alpha^{-1-m})(b'y^j - b'y^i) = 0.
\]

Since \( \overline{B}^{ij} \) are supposed to be \( hp(3) \), the term in (4.5) which seemingly does not contain \( \alpha^2 \) is \( 2(1 - m^2)\beta^{-2-m}\overline{B}^{ij} \) only, and hence we must have \( hp(3 - m) \) \( u_{3-m}^{ij} \) such that

\[
2(1 - m^2)\beta^{-2-m}\overline{B}^{ij} = \alpha^2u_{3-m}^{ij}.
\]

We treat of the general case \( \alpha^2 \neq 0 \) (mod. \( \beta \)). (4.6) shows that there exist \( hp(1) \) \( u^{ij} \) satisfying \( u_{3-m}^{ij} = \beta^{2-m}u^{ij} \). Then (3.4) is reduced to

\[
2(1 - m^2)\beta^{-2-m}\overline{B}^{ij} = \alpha^2u^{ij}.
\]

If \( m \neq 1 \), that is, \( F^n \) is not a Kropina space, then (4.7) gives \( \overline{B}^{ij} \) and (4.5) can be rewritten in the form

\[
\{(1 - m)\beta^2 + mb^2\alpha^2\}\frac{\beta^{-m}u^{ij}}{1 - m} + 2(m\beta^{-1-m} - \alpha^{-1-m})(s'_0y^j - s'_0y^i) - m\{(1 + m)r_{00}\beta^{-m} + 2s_0(ma^2\beta^{-1-m} - \alpha^{-1-m})\}(b'y^j - b'y^i) = 0.
\]

Collecting the terms of (4.8) which seemingly do not contain \( \beta \), we can put

\[
2ma^{-1-m}\{b^2(s'_0y^j - s'_0y^i) - s_0(b'y^j - b'y^i)\} = \beta v_{2-m}^{ij},
\]

where \( v_{2-m}^{ij} \) are \( hp(2 - m) \). Consequently we have

\[
v_{2-m}^{ij} = 2ma^{-1-m}v^{ij} \quad \text{with} \quad hp(1) \quad v^{ij}.
\]

and \( u_{2-m}^{ij} = 2ma^{-1-m}v^{ij} \) with \( hp(1) \) \( v^{ij} \). Thus (4.8) is reduced to

\[
\{(1 - m)\beta^2 + mb^2\alpha^2\}\frac{\beta^{-m}u^{ij}}{1 - m} + 2m^2\alpha^2\beta^{-m}v^{ij} - m\{(1 + m)r_{00}\beta^{-m} - 2s_0a^{-1-m}\}(b'y^j - b'y^i) = 0.
\]
Consequently (4.9) is obtained as follows:

\[(4.11) \quad b^2 s_{ij} = b_i s_j + b_j s_i, \quad \text{provided that} \quad b^2 \neq 0.\]

That is, (4.1). From (4.11), (4.9) is reduced to \(v^{ij} = y^i s^j - y^j s^i\) and (4.10) is rewritten in the form

\[(4.12)\]

\[
\beta^m s_i \left\{ \frac{(1 - m)\beta^2 + mb^2\alpha^2}{1 - m} \left( \frac{\beta^{-m}u^{ij}}{1 - m} - \frac{2(m\beta^{-m} - \alpha^{-1-m}\beta)}{b^2} (s^i y^j - s^j y^i) \right) \right\} \\
+ \left\{ \frac{2m(1 - m)\beta^{1-m} - 2\alpha^{-1-m}((1 - m)\beta^2 + mb^2\alpha^2)}{b^2} \right\} s_0 \\
- m \{(1 + m)r_{00}\alpha^{-1-m} - 2 s_0 \alpha^{-1-m} \} (b^i y^j - b^j y^i) = 0.
\]

Multiplying (4.12) by \(\beta^m\), we obtain

\[(4.13)\]

\[
\beta^m s_i \left\{ \frac{(1 - m)\beta^2 + mb^2\alpha^2}{1 - m} \left( \frac{u^{ij}}{1 - m} - \frac{2(m - \alpha^{-1-m}\beta^{1+m})}{b^2} (s^i y^j - s^j y^i) \right) \right\} \\
+ \left\{ \frac{2m(1 - m)\beta - 2\alpha^{-1-m}\beta^{m}((1 - m)\beta^2 + mb^2\alpha^2)}{b^2} \right\} s_0 \\
- m \{(1 + m)r_{00} - 2 s_0 \alpha^{-1-m} \} (b^i y^j - b^j y^i) = 0.
\]

Transvecting (4.13) by \(b_i s_j\), we have

\[(4.14)\]

\[
\beta^{m} s_i \left\{ \frac{1}{1 - m} u^{ij} b_i s_j + \frac{2}{b^2} (m - \alpha^{-1-m}\beta^{1+m}) s^i s_j \beta \right\} \\
= \left\{ m \{(1 + m)r_{00} - 2 s_0 \alpha^{-1-m} \beta^{m} \} \\
- 2 \left\{ m(1 - m)\beta - \alpha^{-1-m}\beta^{m}((1 - m)\beta^2 + mb^2\alpha^2) \right\} s_0 \right\} b^2 s_0.
\]

Suppose that there exists \(u = u_i(x) y^i\) such that \((1 - m)\beta^2 + mb^2\alpha^2 = b^2 s_0 u\). Then this is written in the form

\[
2\{(1 - m)b_i b_j + mb^2 a_{ij}\} = b^2 (s_i u_j + s_j u_i).
\]

Transvection by \(b^i b^j\) leads to the contradiction \(b^2 = 0\). Therefore (4.14) shows that we have a function \(h_1(x)\) satisfying

\[
\frac{1}{1 - m} u^{ij} b_i s_j + \frac{2}{b^2} (m - \alpha^{-1-m}\beta^{1+m}) s^i s_j \beta = h_1(x) b^2 s_0.
\]
\[
\left\{ \{m(1+m)\alpha 1-m\beta^m\} - 2 \left[ m(1-m)\beta - \alpha^{-1-m}\beta^m. \right. \right. \\
\left. \left. \{(1-m)\beta^2 + mb^2\alpha^2\} \left[ \frac{s_0}{b^2} \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \r
Next, we deal with $m > 1$. Multiplication of (4.3) by $\alpha^{-1+m}$ leads to $s_0 = 0$ and $s_{ij} = 0$. Thus we obtain $r_{00} = h_4(x)\{(1 - m)\beta^2 + mb^2\alpha^2\}$ in common with $s_0 = 0$.

The case (II): Since $\alpha^{1-m}\beta^{m+1}$ is irrational in $(y')$, (4.4) is divided into two equations as follows:

\begin{align*}
(4.19) & \quad 2\{(1 - m)\beta^2 + mb^2\alpha^2\}\{(1 + m)\beta B^{ij} + m\alpha^2(s_j^i y^j - s_j^i y^i)\} \\
& \quad \quad - m\alpha^2\{(1 + m)r_{00} + 2ms_0\alpha^2\}(b^i y^j - b^j y^i) = 0,
\end{align*}

\begin{align*}
(4.20) & \quad \{(1 - m)\beta^2 + mb^2\alpha^2\}(s_j^i y_j - s_j^i y_j) - ms_0\alpha^2(b^j y^i - b^i y^j) = 0.
\end{align*}

Transvecting (4.20) by $b_i y_j$, we get

\[ s_0\alpha^2\{(1 - m)\beta^2 + mb^2\alpha^2\} - ms_0\alpha^2(b^2\alpha^2 - \beta^2) = 0, \]

which implies $s_0\alpha^2\beta = 0$. Hence we get $s_0 = 0$, that is, $s_j = 0$. (4.20) is reduced to $s_j^i y_j - s_j^i y_j = 0$. Transvection of this by $y_i$ leads to $s_0^i = 0$. Therefore $s_{ij} = 0$. Substituting $s_{ij} = 0$ in (4.19), we obtain

\begin{align*}
(4.21) & \quad 2\{(1 - m)\beta^2 + mb^2\alpha^2\}B^{ij} - m\alpha^2r_{00}(b^i y^j - b^j y^i) = 0.
\end{align*}

The term in (4.21) which seemingly does not contain $\alpha^2$ is $2(1 - m)\beta^2B^{ij}$ only, and hence we must have $hp(3) u^{ij}_3$ satisfying

\begin{align*}
(4.22) & \quad 2(1 - m)\beta^2B^{ij} = \alpha^2u^{ij}_3.
\end{align*}

Suppose $\alpha^2 \not\equiv 0 \pmod{\beta}$. Then (4.22) is reduced to $B^{ij} = \alpha^2u^{ij}$, where $u^{ij}$ are $hp(1)$. Hence (4.21) leads to

\begin{align*}
(4.23) & \quad 2\{(1 - m)\beta^2 + mb^2\alpha^2\}u^{ij} - r_{00}(b^i y^j - b^j y^i) = 0.
\end{align*}

Transvecting (4.23) by $b_i y_j$, we obtain

\[ 2\{(1 - m)\beta^2 + mb^2\alpha^2\}u^{ij}b_i y_j - r_{00}(b^2\alpha^2 - \beta^2) = 0. \]

Thus there exists a function $h_4(x)$ such that

\[ 2(m - 1)u^{ij}b_i y_j - r_{00} = h_4(x)\alpha^2, \quad 2mb^2u^{ij}b_i y_j - b^2r_{00} = h_4(x)\beta^2. \]
Eliminating $u^{ij}b_iy_j$ from the above equations, we have

$$b^2r_{00} = h_4(x)((m-1)\beta^2 - mb^2\alpha^2),$$

which implies

$$r_{ij} = \frac{h_4(x)}{b^2}((m-1)b_ib_j - mb^2a_{ij}).$$

From $s_{ij} = 0$ and (4.24) we obtain

$$b_{i;j} = h_5(x)((m-1)b_ib_j - mb^2a_{ij}),$$

where $h_5(x) = h_4(x)/b^2$.

Consequently, if (4.25) is satisfied, then $s_{ij} = 0$ and

$$r_{00} = h_5(x)((m-1)\beta^2 - mb^2\alpha^2),$$

from which $B^{ij}$ of (4.4) are $hp(3)$. Hence (4.18) holds in this case, too.

In any case we obtain $b_{i;j}$ by (4.11) and (4.18), then $B^{ij}$ are given by (4.7) together with (4.16). Consequently a Finsler space $F^n = (M^n, L + \beta)$ ($n > 2$) with non zero $b^4$ which is obtained by Randers change of a generalized Kropina space $F^n = (M^n, L = \alpha^{1+m}\beta^{-m}, m \neq \pm 1, 0)$ is a Douglas space, if and only if $b_{i;j}$ are given by (4.11) and (4.18). That is, (4.1) and (4.2) hold.

On the other hand, it has been known ([8]) that a generalized Kropina space $F^n$ ($n > 2$) with non zero $b^2$ is a Douglas space, if and only if $b_{i;j}$ are given by (4.1) and (4.2). That is to say, the case $s_0 \neq 0$ for $F^n$ to be a Douglas space corresponds to the case $m = -3$ for $F^n$ to be a Douglas space and the case $s_0 = 0$ for $F^n$ to be of Douglas type corresponds to the case $m \neq -3, m \in \mathbb{R}$ for $F^n$ to be of Douglas type. Thus we obtain the following

\textbf{Theorem 4.1.} Let $F^n$ ($n > 2$) be a generalized Kropina space with $L = \alpha^{1+m}\beta^{-m}$, $m$ being a constant $\neq \pm 1, 0$. A Finsler space $F^n$ which is obtained by a special Randers change of $F^n$ with non zero $b^2$ of Douglas type is also of Douglas type, and vice versa.
5. Kropina space

Let $F^n$ be a Kropina space with $L = \alpha^2/\beta$ and $\tilde{F}^n = (M^n, \tilde{L})$ a Finsler space which is obtained by Randers change of $F^n = (M^n, L)$. From (2.4), $\tilde{B}^{ij} = \tilde{B}^i y^j - \tilde{B}_j y^i$ in $\tilde{F}^n$ are written as

$$(5.1) \quad \tilde{B}^{ij} = B^{ij} + \frac{\alpha}{L_\alpha} (s^i_0 y^j - s^j_0 y^i) - \frac{\alpha^4 s_0 L_{\alpha\alpha}}{L_\alpha (\beta^2 L_\alpha + \alpha \gamma^2 L_{\alpha\alpha})} (b^i y^j - b^j y^i).$$

Suppose $F^n$ is a Douglas space. Since $B^{ij}$ are $hp(3)$, the necessary and sufficient condition for $F^n$ to be also a Douglas space is that $W^{ij}$ are $hp(3)$. Thus we have the following

**Proposition 5.1.** Let $F^n = (M^n, L)$ be a Finsler space with an $(\alpha, \beta)$-metric of Douglas type. Then $\tilde{F}^n = (M^n, L + \beta)$ which is obtained by a special Randers change of $F^n$ is also a Douglas space, if and only if $W^{ij}$ are $hp(3)$.

We suppose $F^n$ is a Douglas space. The condition for $\tilde{F}^n = (M^n, L + \beta)$ to be a Douglas space is that (5.2) is $hp(3)$. From (5.2) we have

$$(5.2) \quad W^{ij} = \frac{\alpha}{L_\alpha} (s^i_0 y^j - s^j_0 y^i) - \frac{\alpha^4 s_0 L_{\alpha\alpha}}{L_\alpha (\beta^2 L_\alpha + \alpha \gamma^2 L_{\alpha\alpha})} (b^i y^j - b^j y^i)$$

are $hp(3)$. Thus we have the following

$$(5.3) \quad \tilde{B}^{ij} = \frac{1}{2\beta} (s^i_0 y^j - s^j_0 y^i) - \frac{s_0 \beta}{\beta^2} (b^i y^j - b^j y^i).$$

Since $B^{ij}$ and $W^{ij}$ are $hp(3)$, $\tilde{B}^{ij}$ are $hp(3)$, that is, $\tilde{F}^n$ is a Douglas space. Thus a Kropina space $F^n$ is of Douglas type, then a Finsler space $\tilde{F}^n$ which is obtained by a special Randers change of $F^n$ is of Douglas type also. We consider the condition for a Finsler space which is obtained by a special Randers change of a Kropina space to be of Douglas type. For $\tilde{F}^n = (M^n, \tilde{L} = \alpha^2/\beta)$, (2.4) gives

$$(5.3) \quad \tilde{B}^{ij} = \frac{1}{2\beta} (s^i_0 y^j - s^j_0 y^i) + \frac{1}{2\beta^2} (r_{00} \beta - s_0 (\beta^2 - \alpha^2)) (b^i y^j - b^j y^i).$$
Since the terms \((\beta/2)(s^i_0y^j - s^j_0y^i) + (1/2\beta^2\beta)(r_{00} - s_0\beta)(b^i_0y^j - \hat{b}^j_0y^i)\)
are \(h\mathcal{P}(3)\), these terms may be neglected in our discussion and we treat
only of \(5.5\)

\[
W^{ij} = \alpha^2 \left\{ \frac{s_0}{\mathcal{A}}(b^i_0y^j - b^j_0y^i) - (s^i_0y^j - s^j_0y^i) \right\}.
\]

For \(n > 2\), \(\alpha^2 \not\equiv 0 \pmod{\beta} \) \([3]\). Therefore there exist \(h\mathcal{P}(1) v^{ij} = v^{ij}_k(x)y^k\) such that
\(5.6\)

\[
\frac{s_0}{\mathcal{A}}(b^i_0y^j - b^j_0y^i) - (s^i_0y^j - s^j_0y^i) = \beta v^{ij}.
\]

This equation is written in the form
\(5.7\)

\[
\frac{1}{\mathcal{A}}b^i(s^j_\delta^k + s_k^\delta^j_\delta^i - b^j(s^k_\delta^i + s_k^\delta^j_\delta^i))
-(s^j_\delta^i + s_k^\delta^j_\delta^i) + (s^j_\delta^i + s^k_\delta^i_\delta^j) = b_k v^{ij}.
\]

Transvection of \(5.7\) by \(a^hk\) leads to
\(5.8\)

\[
\frac{1}{\mathcal{A}}b^i(s^j_\delta^i + b^j s^k) - 2s^{ij} = b^i v^{ij}.
\]

Next, transvecting \(5.7\) by \(b^h\), we have
\(5.9\)

\[
(s^j_\delta^i + b^j s^k) - (s^i_\delta^j + b^i s^k) = b^2 v^{ij}_k + b_k b^i v^{ij}.
\]

Contraction of \(5.7\) with \(j\) and \(h\) leads to
\(5.10\)

\[
n \left( \frac{1}{\mathcal{A}}b^i s^k - s^i_k \right) = b_r v^{ir}_k - b_k v^{ir}.
\]

Substituting \(b^i v^{ij}_k\) of \(4.8\) in \(4.9\), we have
\[
b^2 v^{ij}_k = 2s^{ij}b_k + \left\{ b^i s^j_k - b^j s^i_k + s^i_\delta^j_k - s^j_\delta^i_k + \frac{1}{\mathcal{A}}(s^i b^j b_k - s^j b^i b_k) \right\},
\]

which imply
\[
b^2 v^{i'j'} = (n - 1)s^i, \quad b^2 b_r v^{ir}_k = b^i s_k - b^i s^i_k.
\]
Consequently (5.10) leads to

\[(5.11) \quad s_{ij} = \frac{1}{b^2} (b_is_j - b_js_i).\]

Then (5.5) gives

\[W_{ij} = \frac{\alpha^2}{2b^2} (s^i y^j - s^j y^i),\]

which are hp(3). Therefore (5.11) is the necessary and sufficient condition for \(F^n\) to be of Douglas type.

On the other hand, it is known ([8]) that a Kropina space \(F^n(n > 2)\) with \(b^2 \neq 0\) is of Douglas type, if and only if (5.11) is satisfied. Thus we have the

**Theorem 5.2.** A Finsler space \(F^n(n > 2)\) which is obtained by a special Randers change of a Kropina space \(F^n\) with \(b^2 \neq 0\) is of Douglas type, if and only if the Kropina space \(F^n\) is of Douglas type.

### 6. Matsumoto space

We consider the condition for a Finsler space \(F^n = (M^n, L + \beta)\) which is obtained by a special Randers change of Matsumoto space \(F^n = (M^n, L = \alpha^2/(\alpha - \beta))\) to be of Douglas type. It is known ([6]) that a Matsumoto space \(F^n(n > 2)\) is of Douglas type, if and only if \(b_{i;j} = 0\). Hence, for a Matsumoto space \(F^n\) of Douglas type, (2.4) leads to \(\overline{W}^{ij} = 0\), that is, \(\overline{B}^{ij} = B^{ij}\). Thus if a Matsumoto space \(F^n\) is of Douglas type, then a Finsler space which is obtained by a special Randers change of \(F^n\) is also of Douglas type. It is known ([8]) that a Matsumoto space \(F^n(n > 2)\) is of Douglas type, if and only if \(b_{i;j} = 0\). Hence, for a Matsumoto space \(F^n\) of Douglas type, (5.2) leads to \(\overline{W}^{ij} = 0\), that is, \(\overline{B}^{ij} = B^{ij}\). Thus if a Matsumoto space \(F^n\) is of Douglas type, then a Finsler space which is obtained by a special Randers change of \(F^n\) is also of Douglas type. For \(F^n\), (2.3) gives

\[(6.1) \quad \{\alpha(1 + 2\beta - 3\beta)\{(\alpha - 2\beta)\overline{B}^{ij} - (2\alpha^2 - 2\alpha\beta + \beta^2)(s^i y^j - s^j y^i)\} \]
\[+ \alpha \{2s_0(2\alpha^2 - 2\alpha\beta + \beta^2) - r_{00}(\alpha - 2\beta)\} (b_i^iy^j - b_j^iy^i) = 0.\]

Suppose that \(F^n\) be a Douglas space, that is, \(\overline{B}^{ij}\) be hp(3). Since \(\alpha\)
is irrational in \((y')\), (6.1) is divided as follows:

\[
[(1 + 2b^2)\alpha^2 + 6\beta^2]B^ij + \begin{align*}
&\{2\alpha^2\beta(1 + 2b^2) + 3\beta(2\alpha^2 + \beta^2)\}(s'^0y^j - s'^0y'^j) \\
&\quad - (4s_0\alpha^2\beta + r_{00}\alpha^2)(b^iy^j - b^iy'^j) = 0,
\end{align*}
\]

\[
(5 + 4b^2)\beta B^ij + \begin{align*}
&\{(1 + 2b^2)(2\alpha^2 + \beta^2) + 6\beta^2\}(s'^0y^j - s'^0y'^j) \\
&\quad - 2\{s_0(2\alpha^2 + \beta^2) + r_{00}\beta\} (b^iy^j - b^iy'^j) = 0.
\end{align*}
\]

Eliminating \(B^ij\) from these equations, we have

\[
A(s'^0y^j - s'^0y'^j) + B(b^iy^j - b^iy'^j) = 0,
\]

where we put

\[
A = \alpha^2(21\beta^2 + 12\beta^2b^2 + 12\beta^2b^4 - 2\alpha^2 - 8\alpha^2b^2 - 8\alpha^2b^4) - 27\beta^4,
\]

\[
B = \alpha^2\{s_0(6\beta^2 - 12\beta^2b^2 + 4\alpha^2 + 8\alpha^2b^2) - 3r_{00}\beta\} + 12\beta^3(s_0\beta + r_{00}).
\]

Transvection of (6.4) by \(b_iy_j\) leads to

\[
As_0\alpha^2 + B(b^2\alpha^2 - \beta^2) = 0.
\]

Since the terms \(12(s_0\beta + r_{00})\beta^5\) of (6.5) seemingly do not contain \(\alpha^2\), we must have \(hp(5)\) \(v_5\) such that

\[
12(s_0\beta + r_{00})\beta^5 = \alpha^2v_5.
\]

In the first case of \(v_5 = 0\), we have \(r_{00} = -s_0\beta\) from (6.6), and (6.5) is reduced to

\[
\{\alpha^2(17\beta^2 + 13\beta^2b^2 - 2\alpha^2 - 4\alpha^2b^2) + 12\beta^4(b^2 - 3)\}s_0 = 0.
\]

If the coefficient of \(s_0\) does not vanish, then

\[
\alpha^2(17\beta^2 + 13\beta^2b^2 - 2\alpha^2 - 4\alpha^2b^2) = 12\beta^4(3 - b^2).
\]

Since we suppose \(\alpha^2 \not\equiv 0 \pmod{\beta}\), the above assumption is a contradiction. Therefore we obtain \(s_0 = 0\) and \(r_{00} = 0\) from (6.6). Next, in the second case of \(v_5 \neq 0\), (6.6) shows the existence of a function
Douglas space, then \( L = \frac{\alpha^2}{\beta} \), and hence \( r_{00} = k_2(x)\alpha^2 - s_0\beta \), where \( k_2(x) = k_1(x)/12 \). Then (6.5) is reduced to

\[
A_{s_0} + \{ s_0(9\beta^2 - 12\beta^2 b^2 + 4\alpha^2 + 8\alpha^2 b^2) - 3k_2(x)\beta(\alpha^2 - 4\beta^2) \}(b^2\alpha^2 - \beta^2) = 0.
\]

Only the terms \(-36s_0\beta^4 + 12\beta^4 b^2 s_0 - 12k_2(x)\beta^5\) of (6.7) seemingly do not contain \( \alpha^2 \), and hence we must have \( hp(3) \) such that

\[
12\{ s_0(b^2 - 3) - k_2(x)\beta \} \beta^4 = \alpha^2 v_3.
\]

From \( \alpha^2 \not\equiv 0 \) (mod. \( \beta \)) it follows that \( v_3 \) must vanish, and hence \( s_0(b^2 - 3) = k_2(x)\beta \), that is, \( (b^2 - 3)s_i = k_2(x)h_i \). Then transvection by \( b^i \) gives \( k_2(x)b^2 = 0 \). In case of \( k_2(x) = 0 \), we get \( b^2 = 3 \) or \( s_i = 0 \). If \( b^2 = 3 \), then (6.7) is reduced to \( 14s_0(4\beta^2 - \alpha^2)\alpha^2 = 0 \). Thus we obtain \( s_0 = 0 \) and \( r_{00} = 0 \). Next, if \( s_i = 0 \), then we have \( s_0 = 0 \) and \( r_{00} = 0 \), too. On the other hand, in the case of \( b^2 = 0 \), (6.7) is reduced to \( s_0(17\alpha^2\beta^2 - 2\alpha^4 - 36\beta^4) + 3k_2(x)\beta^3(\alpha^2 - 4\beta^2) = 0 \), which implies \( s_0 = 0 \) and \( k_2(x) = 0 \). Therefore, for \( n > 2 \), both the cases of \( v_5 = 0 \) and \( v_5 \neq 0 \) lead to \( r_{00} = 0 \) and \( s_0 = 0 \). Hence (6.4) is reduced to \( s'_{a0}y^2 - s'_{a0}y^1 = 0 \), and transvection by \( y_i \) gives \( s' = 0 \). Finally \( r_{ij} = s_{ij} = 0 \), that is, \( b_{ij} = 0 \).

Thus a Finsler space \( \mathcal{F}^n = (M^n, L + \beta) \) (\( n > 2 \)) which is obtained by a special Randers change of a Matsumoto space \( F^n = (M^n, L = \alpha^2/(\alpha - \beta)) \) is Douglas space, if and only if \( b_{i,j} = 0 \). On the other hand, M. Matsumoto proved ([8]) that a Matsumoto space \( F^n \) (\( n > 2 \)) is of Douglas type, if and only if \( b_{i,j} = 0 \). Thus we have the following

**Theorem 6.1.** A Finsler space \( \mathcal{F}^n \) (\( n > 2 \)) which is obtained by a special Randers change of a Matsumoto space \( F^n \) of Douglas type is also of Douglas type, and vice versa.

On the other hand, it has been shown ([11]) that Matsumoto space is a Berwald space, if and only if \( b_{i,j} = 0 \). Then according to Theorem 6.1 we have the following

**Corollary 6.2.** Let \( \mathcal{F}^n \) (\( n > 2 \)) be a Finsler space which is obtained by a special Randers change of a Matsumoto space \( F^n \). If \( F^n \) is a Douglas space, then \( \mathcal{F}^n \) is a Berwald space.

**7. Finsler space with** \( L = \alpha + \beta^2/\alpha \)

We consider a Finsler space \( F^n = (M^n, L) \) with an \( (\alpha, \beta) \)-metric \( L = \alpha + \beta^2/\alpha \). This metric may be regarded as constructed from \( \alpha \) and
one more Riemannian metric $\sqrt{\alpha^2 + \beta^2}$, and it is thought of as desirable in the viewpoint of geometry and applications ([8]). For $\tilde{F}^n = (M^n, \tilde{L})$ which is obtained by a special Randers change of $F^n = (M^n, L = \alpha + \beta^2/\alpha)$, (2.3) gives

$$\tilde{B}^{ij} = \frac{\alpha^2(\alpha + 2\beta)}{(\alpha^2 - \beta^2)}(s_i^j y^i - s^i_0 y^i) $$

\[ (7.1) \]

\[ + \frac{\alpha^2 r_{00}(\alpha^2 - \beta^2) - 2s_0\alpha^2(\alpha + 2\beta)}{(\alpha^2 - \beta^2)(\alpha^2(1 + 2\beta) - 3\beta^2)}(b^i y^j - b^j y^i) \].

Suppose that $\tilde{F}^n$ be a Douglas space, that is, $\tilde{B}^{ij}$ be $hp$ (3). Separating (7.1) into the rational and irrational terms of $y^i$, we have

$$\{\alpha^2(1 + 2b^2) - 3\beta^2\}\{\alpha^2 - \beta^2\}\tilde{B}^{ij} - 2\alpha^2\beta(s_i^j y^i - s^i_0 y^i)$$

\[- \alpha^2 r_{00}(\alpha^2 - \beta^2) - 4s_0\alpha^2\beta}(b^i y^j - b^j y^i) + \alpha[2s_0\alpha^2(b^i y^j - b^j y^i) - \alpha^2(1 + 2b^2) - 3\beta^2](s_i^j y^i - s^i_0 y^i) = 0,$$

which yield two equations as follows:

\[ (7.2) \]

\[ \{\alpha^2(1 + 2b^2) - 3\beta^2\}\{\alpha^2 - \beta^2\}\tilde{B}^{ij} - 2\alpha^2\beta(s_i^j y^i - s^i_0 y^i)$$

\[- \alpha^2 r_{00}(\alpha^2 - \beta^2) - 4s_0\alpha^2\beta}(b^i y^j - b^j y^i) = 0,$$

\[ (7.3) \]

\[ 2s_0\alpha^2[b^i y^j - b^j y^i] - \{\alpha^2(1 + 2b^2) - 3\beta^2\}(s_i^j y^i - s^i_0 y^i) = 0.$$

Transvecting (7.3) by $b_i y_j$, we obtain

$$2s_0\alpha^2(b^2\alpha^2 - \beta^2) - \{\alpha^2(1 + 2b^2) - 3\beta^2\} s_0 \alpha^2 = 0,$$

which implies $s_0\alpha^2(\beta^2 - \alpha^2) = 0$. Therefore we get $s_i = 0$. Hence (7.3) is reduced to $s_i^j y^i - s^i_0 y^i = 0$, and transvection by $y_i$ gives $s^i_0 = 0$. Consequently $s_{ij} = 0$. On the other hand, substituting (7.3) in (7.2), we have

\[ (7.4) \]

$$\{\alpha^2(1 + 2b^2) - 3\beta^2\}\tilde{B}^{ij} - \alpha^2 r_{00}(b^i y^j - b^j y^i) = 0,$$

Only the terms $3\beta^2\tilde{B}^{ij}$ of (7.4) seemingly do not contain $\alpha^2$. Hence we must have $hp(3) v^i_3$ satisfying

\[ (7.5) \]

$$3\beta^2 \tilde{B}^{ij} = \alpha^2 v^i_3.$$
For the sake of brevity we suppose $\alpha^2 \not\equiv 0 \pmod{\beta}$. Then (7.5) is reduced to $B_{ij} = \alpha^2 v_{ij}$, where $v_{ij}$ are $hP(1)$. Thus (7.4) leads to

$$\{\alpha^2(1 + 2b^2) - 3\beta^2\}v_{ij} = r_{00}(b^iy^j - b^jy^i) = 0.$$  

Transvecting (6.6) by $b_iy_j$, we get

$$\{\alpha^2(1 + 2b^2)b_i v^{ij}y_j - r_{00}(b^2\alpha^2 - \beta^2) = 0,$$

which imply

$$\alpha^2\{(1 + 2b^2)b_i v^{ij}y_j - b^2r_{00}\} = \beta^2(3b_i v^{ij}y_j - r_{00}).$$

Therefore there exists a function $f_1(x)$ satisfying

$$(1 + 2b^2)b_i v^{ij}y_j - b^2r_{00} = f_1(x)\beta^2, \quad 3b_i v^{ij}y_j - r_{00} = f_1(x)\alpha^2.$$ 

Eliminating $b_i v^{ij}y_j$ from above the equations, we obtain

$$r_{00} = f_1(x)\frac{(1 + 2b^2)\alpha^2 - 3\beta^2}{b^2 - 1}.$$ 

From (7.7) and $s_{ij} = 0$,

$$b_{i,j} = f_2(x)\{(1 + 2b^2)a_{ij} - 3b_i b_j\},$$

where $f_2(x) = f_1(x)/(b^2 - 1)$.

Conversely, if (7.8) is satisfied, then $s_{ij} = 0$ and

$$r_{00} = f_2(x)\{(1 + 2b^2)\alpha^2 - 3\beta^2\},$$

from which $\overline{B}_{ij}$ of (7.1) are $hP(3)$. Thus we have the following

**Theorem 7.1.** A Finsler space $F^n$ $(n > 2)$ which is obtained by a special Randers change of a Finsler space $F^n$ with an $(\alpha, \beta)$-metric $L = \alpha + \beta^2/\alpha$ $(b^2 \neq 1)$ of Douglas type, is also a Douglas space, and vice versa.
References


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