ON THE FEKETE-SZEGŐ PROBLEM AND ARGUMENT INEQUALITY FOR STRONGLY QUASI-CONVEX FUNCTIONS

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Abstract. Let $Q(\beta)$ be the class of normalized strongly quasi-convex functions of order $\beta$ in the open unit disk. Sharp Fekete-Szegő inequalities are obtained for functions belonging to the class $Q(\beta)$. We also consider the integral preserving properties in a sector.

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $U = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}$ and let $S$ be the subclass of $\mathcal{A}$ consisting of all univalent functions. We also denote by $S^*, K$ and $C$ the subclasses of $\mathcal{A}$ consisting of functions which are, respectively, starlike, convex and close-to-convex in $U$ (see, e.g., Srivastava and Owa [17]).

For analytic functions $g$ and $h$ with $g(0) = h(0)$, $g$ is said to be subordinate to $h$ if there exists an analytic function $w(z)$ such that $w(0) = 0$, $|w(z)| < 1$ ($z \in U$), and $g(z) = h(w(z))$. We denote this subordination by $g \prec h$ or $g(z) \prec h(z)$.

A classical result of Fekete and Szegő [5] determines the maximum value of $|a_3 - \mu a_2^2|$, as a function of the real parameter $\mu$, for functions...

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belonging to $S$. There are now several results of this type in the literature, each of them dealing with $|a_3 - \mu a_2^2|$ for various classes of functions (see, e.g., [1, 7, 9]).

Denote by $Q(\beta)$ the class of strongly quasi-convex functions of order $\beta (\beta \geq 0)$. Thus $f \in Q(\beta)$ if and only if there exists $g \in K$ such that for $z \in \mathcal{U}$,

$$\left| \arg \left\{ \frac{zf'(z)}{g'(z)} \right\} \right| \leq \frac{\pi}{2} \beta.$$ 

In particular, $Q(1)$ is the class of quasi-convex functions introduced by Noor [13]. We also note that every quasi-convex function is close-to-convex and hence univalent in $\mathcal{U}$.

In the present paper, we derive sharp Fekete-Szegő inequalities for functions belonging to the class $Q(\beta)$. Furthermore, the integral preserving properties are considered for functions in the class $Q(\beta)$.

2. Results

To prove our main results, we need the following lemmas.

**Lemma 2.1.** Let $p$ be analytic in $\mathcal{U}$ and satisfy $\text{Re} \{p(z)\} > 0$ for $z \in \mathcal{U}$, with $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$. Then

$$(2.1) \quad |p_n| \leq 2 \quad (n \geq 1)$$

and

$$(2.2) \quad \left| p_2 - \frac{p_1^2}{2} \right| \leq 2 - \frac{|p_1|^2}{2}.$$ 

**Lemma 2.2.** Let $h$ be convex (univalent) function in $\mathcal{U}$ and $\omega$ be an analytic function in $\mathcal{U}$ with $\text{Re} \{\omega(z)\} \geq 0$. If $p$ is analytic in $\mathcal{U}$ and $p(0) = h(0)$, then

$$p(z) + \omega(z)zp'(z) \prec h(z) \quad (z \in \mathcal{U})$$

implies

$$p(z) \prec h(z) \quad (z \in \mathcal{U}).$$
Lemma 2.3. Let $p$ be analytic in $U$ with $p(0) = 1$ and $p(z) \neq 0$ in $U$. Suppose that there exists a point $z_0 \in U$ such that

\begin{align}
\left| \arg \{p(z)\} \right| &< \frac{\pi}{2} \eta \quad \text{for } |z| < |z_0| \\
\left| \arg \{p(z_0)\} \right| &= \frac{\pi}{2} \eta (0 < \eta \leq 1).
\end{align}

Then

\begin{align}
\frac{z_0 p'(z_0)}{p(z_0)} &= ik\eta,
\end{align}

where

\begin{align}
k &\geq \frac{1}{2} \left( a + \frac{1}{a} \right) \text{ when } \arg \{p(z_0)\} = \frac{\pi}{2} \eta, \\
&\leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \text{ when } \arg \{p(z_0)\} = -\frac{\pi}{2} \eta,
\end{align}

and

\begin{align}
(p(z_0))^\frac{1}{a} &= \pm ia \quad (a > 0).
\end{align}

The inequality (2.1) was first proved by Carathéodory [3] (also, see Duren [4, p. 41]) and the inequality (2.2) can be found in [15, p. 166]. Lemma 2.2 are the result proved by Miller and Mocanu [11], which has a number of important applications in the theory of univalent functions. Also Lemma 2.3 was proved by Nunokawa [14] as a new modification of well known Jack’s Lemma [6].

With the help of Lemma 2.1, we now derive

Theorem 2.1. Let $f \in \mathcal{Q}(\beta)$ and be given by (1.1). Then for $\beta \geq 0$, we have

\begin{align}
9 \left| a_3 - \mu a_2^2 \right| &\leq \begin{cases}
1 + \frac{(1+\beta)^2(8-9\mu)}{4} & \text{if } \mu \leq \frac{8\beta}{9(1+\beta)}, \\
1 + 2\beta + \frac{2(8-9\mu)}{8-\beta(8-9\mu)} & \text{if } \frac{8\beta}{9(1+\beta)} \leq \mu \leq \frac{8}{9}, \\
1 + 2\beta & \text{if } \frac{8}{9} \leq \mu \leq \frac{8(2+\beta)}{9(1+\beta)}, \\
-1 + \frac{(1+\beta)^2(9\mu-8)}{4} & \text{if } \mu \geq \frac{8(2+\beta)}{9(1+\beta)}.
\end{cases}
\end{align}
For each $\mu$, there is a function in $Q(\beta)$ such that equality holds in all cases.

Proof. Let $f \in Q(\beta)$. Then it follows from the definition that we may write

$$
(2.9) \quad \frac{(zf'(z))'}{g'(z)} = p^\beta(z),
$$

where $g$ is convex and $p$ has positive real part. Let $g(z) = z + b_2 z^2 + b_3 z^3 + \cdots$ and let $p$ be given as in Lemma 2.1. Then by comparing the coefficients of both sides of (2.9), we obtain

$$
4a_2 = \beta p_1 + 2b_2
$$

and

$$
9a_3 = \frac{\beta(\beta - 1)}{2} p_1^2 + \beta p_2 + 3b_3 + 2\beta p_1 b_2.
$$

So, with $x = (8 - 9\mu)/4$, we have

$$
9(a_3 - \mu a_2^2) = 3 \left( b_3 + \frac{1}{3}(x - 2)b_2^2 \right)
\quad + \beta \left( p_2 + \frac{1}{4}(\beta x - 2)p_1^2 \right) + \beta xp_1 b_2.
$$

Since rotations of $f$ also belong to $Q(\beta)$, without loss of generality, we may assume that $a_3 - \mu a_2^2$ is positive. Thus we now estimate $\text{Re} \ (a_3 - \mu a_2^2)$.

Since $g \in K$, there exists $h(z) = 1 + k_1 z + k_2 z^2 + \cdots \ (z \in \mathcal{U})$ with positive real part, such that $g'(z) + zg''(z) = g'(z) h(z)$. Hence, by equating coefficients, we get that $b_2 = k_1/2$ and $b_3 = (k_2 + k_1^2)/6$. Therefore, letting $b_2 = \rho e^{i\phi} (0 \leq \rho \leq 1)$ and $p_1 = 2re^{i\theta} (0 \leq r \leq 1)$ in (2.10), and applying Lemma 2.1, we obtain

$$
9\text{Re}(a_3 - \mu a_2^2) \leq (1 - \rho^2) + (x + 1)\rho^2 \cos 2\phi
\quad + 2\beta(1 - r^2) + \beta^2 r^2 \cos 2\theta + 2\beta x r \cos(\theta + \phi)
\quad = \psi(x), \ \text{say}.
$$
We consider first the case \( 8 \beta/(9(1 + \beta)) \leq \mu \leq 8/9 \). In this case, we see that \( 0 \leq x \leq 2/(1 + \beta) \). Then we obtain
\[
\begin{align*}
\psi(x) &= 1 - \rho^2 + (x + 1)\rho^2 \cos 2\phi + \beta(2(1 - r^2) + \beta x r^2 \cos 2\theta \\
&\quad + 2r \rho \cos(\theta + \phi)) \\
&\leq x + 1 + \beta(2 - 2r^2 + \beta x r^2 \cos 2\theta + 2r).
\end{align*}
\]
Since the expression \(-2t^2 + \beta xt^2 \cos 2\theta + 2xt\) is the largest when \( t = x/(2 - \beta x \cos 2\theta) \), we have
\[
-2t^2 + \beta xt^2 \cos 2\theta + 2xt \leq \frac{x^2}{2 - \beta x \cos 2\theta} \leq \frac{x^2}{2 - \beta x}.
\]
Thus
\[
\psi(x) \leq x + 1 + \beta \left(2 + \frac{x^2}{2 - \beta x} \right)
= 1 + 2\beta + \frac{2(8 - 9\mu)}{8 - \beta(8 - 9\mu)},
\]
and from (2.11), we obtain the second inequality of the theorem. Equality occurs only if
\[
p_1 = \frac{2(8 - 9\mu)}{8 - \beta(8 - 9\mu)}, \quad p_2 = 2, \quad b_2 = b_3 = 1,
\]
and the corresponding function \( f \) is defined by
\[
(zf'(z))' = \frac{1}{(1 - z)^2} \left(\frac{\lambda}{1 - z} + (1 - \lambda)\frac{1 - z}{1 + z}\right)^\beta, \quad f(0) = 0,
\]
where
\[
\lambda = \frac{8 + (1 - \beta)(8 - 9\mu)}{16 - 2\beta(8 - 9\mu)}.
\]

We now prove the first inequality. Let \( \mu \leq 8\beta/(9(1 + \beta)) \). Then we obtain that \( x \geq 2/(1 + \beta) = x_0 \), and
\[
\psi(x) = \psi(x_0) + (x - x_0)(\rho^2 \cos 2\phi + \beta^2 r^2 \cos 2\theta + 2\beta r \rho \cos(\theta + \phi)) \\
\leq \psi(x_0) + (x - x_0)(1 + \beta)^2 \\
\leq 1 + \frac{(1 + \beta)^2(8 - 9\mu)}{4}.
\]
as required. Equality occurs only if $c_1 = c_2 = 2, \ b_2 = b_3 = 1$, and the corresponding function $f$ is defined by

$$(zf'(z))' = \frac{1}{(1-z)^2} \left( \frac{1+z}{1-z} \right)^\beta, \ f(0) = 0.$$ 

Let $x_1 = -2/(1 + \beta)$. At first, we will show that $\psi(x_1) \leq 1 + 2\beta$. Then the remaining inequalities follow easily from this one. We have

$$(-2 + \beta x_1 \cos 2\theta)t^2 + 2x_1 t \rho \cos(\theta + \phi) \leq \frac{x_1^2 \rho^2 \cos^2(\theta + \phi)}{2 - \beta x_1 \cos 2\theta}$$

for all real $t$. Hence we obtain

$$\psi(x_1) - (1 + 2\beta) \leq \rho^2 \left(-1 + (x_1 + 1) \cos 2\phi + \frac{\beta x_1^2 (1 + \cos 2(\theta + \phi))}{2(2 - \beta x_1 \cos 2\theta)} \right).$$

Thus we consider the inequality

$$\beta x_1^2 (1 + \cos 2(\theta + \phi)) + 2(2 - \beta x_1 \cos 2\theta)(-1 + (x_1 + 1) \cos 2\phi) \leq 0,$$

which is true if

$$(2.12) \quad 2\beta^2 \cos^2 \theta \sin^2 \phi + 2\beta \cos \theta \sin \phi \cos \phi + \cos^2 \phi \geq 0.$$

Now, for all real $t$

$$2t^2 + 2t \sin \theta \cos \phi + \cos^2 \phi \geq 0,$$

so, by taking $t = \beta \cos \theta \sin \phi$, we obtain (2.12). Thus $\psi(x_1) \leq 1 + 2\beta$.

Next, we consider two possibilities. We suppose that $x_1 \leq x \leq 0$, that is, $8/9 \leq \mu \leq 8(2 + \beta)/(9(1 + \beta))$. Note that for $0 \leq \lambda \leq 1$,

$$\psi(\lambda x_1) = \lambda \psi(x_1) + (1 - \lambda)\psi(0) = 1 + 2\beta.$$

Hence we have $\psi(x) \leq 1 + 2\beta$ and this proves the third inequality of the theorem. Equality occurs only if $p_1 = b_2 = 0, \ p_2 = 2, \ b_3 = 1/3$, and the corresponding function $f$ is defined by

$$(zf'(z))' = \frac{(1 + z^2)^\beta}{(1 - z^2)^{1+\beta}}, \ f(0) = 0.$$
Secondly, we suppose that $x \leq x_1$, that is, $\mu \geq \frac{8(2 + \beta)}{9(1 + \beta)}$.
Then we have
\[
\psi(x_0) = \psi(x_1) + (x - x_1)(\rho^2 \cos 2\phi + \beta^2 r^2 \cos 2\theta + 2\beta \rho r \cos(\theta + \phi)) \\
\leq \psi(x_1) + (x_1 - x)(1 + \beta)^2 \\
\leq -1 + \frac{(1 + \beta)^2(9\mu - 8)}{4},
\]
and this is the last inequality of the theorem. Equality occurs only if $p_1 = 2i, \ p_2 = -2, \ b_2 = i, \ b_3 = -1$, and the corresponding function $f$ is defined by
\[
(zf'(z))' = \frac{1}{(1 - iz)^2} \left(\frac{1 + iz}{1 - iz}\right)^\beta, \ f(0) = 0.
\]
Therefore we complete the proof of Theorem 2.1.

For a function $f$ belonging to the class $A$, we define the integral operator $F_\gamma$ as follows :
\[
(2.13) \quad F_\gamma(f) := F_\gamma(f)(z) = \frac{\gamma + 1}{z^\gamma} \int_0^z t^{\gamma - 1} f(t) dt \ (\gamma \geq 0 ; \ z \in \mathcal{U}).
\]
Many authors have studied the integral operator of the form (2.13) where $\gamma$ is a real constant and $f$ belongs to some favored classes of functions. Various interesting developments involving the operator (2.13), for examples, can be found in [2, 8, 10]. We also denote the class $K[A, B]$ by
\[
K[A, B] = \{ f \in A : 1 + \frac{zf''(z)}{f'(z)} < \frac{1 + Az}{1 + Bz} \ (z \in \mathcal{U} ; -1 \leq B < A \leq 1) \}.
\]
Next, we prove

**Theorem 2.2.** Let $f \in A$. If
\[
\left| \arg \left\{ \frac{(zf'(z))'}{g'(z)} \right\} \right| < \frac{\pi}{2} \delta \ (0 < \delta \leq 1 ; \ z \in \mathcal{U})
\]
for some $g \in K[A, B]$, then
\[
\left| \arg \left\{ \frac{(ZF_\gamma(f))'}{F_\gamma(g)} \right\} \right| < \frac{\pi}{2} \eta.
\]
where $F_\gamma$ is given by (2.13) and $\eta(0 < \eta \leq 1)$ is the solution of the equation:

(2.14)  
$$
\delta = \begin{cases} 
\eta + \frac{2}{\pi} \tan^{-1} \left( \frac{\eta \sin \frac{\pi}{2} (1-t(A,B,c))}{(1+2+\pi) \eta \cos \frac{\pi}{2} (1-t(A,B,c))} \right) & \text{for } B \neq -1, \\
\eta & \text{for } B = -1, 
\end{cases}
$$

when

(2.15)  
$$
t(A, B, c) = \frac{2}{\pi} \sin^{-1} \left( \frac{A-B}{1-AB+c(1-B^2)} \right).$$

**Proof.** Let

$$p(z) = \frac{z(F'(f))'}{F'(g)} \quad \text{and} \quad q(z) = 1 + \frac{zF''(g)}{F'(g)}.$$

From the assumption for $g$ and an application of Briot-Bouquet differential subordination [12, p. 81], we see that $F_\gamma(g) \in \mathcal{K}[A, B]$. Using the equation

$$zF'_\gamma(f)(z) + \gamma F_\gamma(f)(z) = (1 + \gamma)f(z)$$

and simplifying, we obtain

$$\left( \frac{zf'(z)}{g'(z)} \right)' = p(z) + \frac{zp'(z)}{q(z) + c}.$$

Since $q \in \mathcal{K}[A, B]$, we note [16] that

(2.16)  
$$\left| \frac{(zF'_\gamma(g))'}{F'_\gamma(g)} - \frac{1-AB}{1-B^2} \right| < \frac{A-B}{1-B^2} \quad (z \in \mathcal{U} ; \ B \neq -1)$$

and

(2.17)  
$$\text{Re} \left\{ \frac{(zF'_\gamma(g))'}{F'_\gamma(g)} \right\} > \frac{1-A}{2} \quad (z \in \mathcal{U} ; \ B = -1).$$

Then, from (2.16) and (2.17), we have

$$q(z) + c = \rho e^{i \frac{\pi}{2} \phi},$$
where
\[
\begin{cases}
\frac{1-A}{2} + c < \rho < \frac{1+A}{2} + c \\
-t(A, B, c) < \phi < t(A, B, c)
\end{cases}
\text{for } B \neq -1,
\]
where \(t(A, B, c)\) is given by (2.15), and
\[
\begin{cases}
\frac{1-A}{2} + c < \rho < \infty \\
-1 < \phi < 1
\end{cases}
\text{for } B = -1.
\]

Here, we note that \(p\) is analytic in \(U\) with \(p(0) = 1\) and \(\text{Re } p(z) > 0\) in \(U\) by applying the assumption and Lemma 2.2 with \(\omega(z) = 1/(q(z) + c)\). Hence \(p(z) \neq 0\) in \(U\).

If there exists a point \(z_0 \in U\) such that the conditions (2.3) and (2.4) are satisfied, then (by Lemma 2.3) we obtain (2.5) under the restrictions (2.6-8).

At first, we suppose that
\[
\{p(z_0)\}^\frac{1}{k} = ia \quad (a > 0).
\]

For the case \(B \neq -1\), we then obtain
\[
\begin{align*}
\arg \left\{ \frac{(z_0 f'(z_0))'}{g'(z_0)} \right\} &= \arg \left\{ p(z_0) \left( 1 + \frac{1}{q(z_0) + c \cdot p(z_0)} \right) \right\} \\
&= \arg \{ p(z_0) \} + \arg \left\{ 1 + i\eta k (p(e^{i\frac{\pi}{2}} - 1) \right\} \\
&= \frac{\pi}{2} \eta + \tan^{-1} \left( \frac{\eta k \sin \left( \frac{\pi}{2} - \phi \right)}{\rho + \eta k \cos \left( \frac{\pi}{2} - \phi \right)} \right) \\
&\geq \frac{\pi}{2} \eta + \tan^{-1} \left( \frac{\eta \sin \left( \frac{\pi}{2} (1 - t(A, B)) \right)}{\left( \frac{1+A}{2} + c \right) + \eta \cos \left( \frac{\pi}{2} (1 - t(A, B)) \right) \right) \\
&= \frac{\pi}{2} \delta,
\end{align*}
\]
where \(\delta\) and \(t(A, B)\) are given by (2.14) and (2.15), respectively. Similarly, for the case \(B = -1\), we have
\[
\begin{align*}
\arg \left\{ \frac{(z_0 f'(z_0))'}{g'(z_0)} \right\} &\geq \frac{\pi}{2} \eta \geq \frac{\pi}{2} \delta.
\end{align*}
\]
These evidently contradict the assumption of the theorem.

Next, in the case \( p(z_0)^{1/2} = -ia \ (a > 0) \), applying the same method as the above, we also can prove the theorem easily. Therefore we complete the proof of Theorem 2.2.

**Remark.** From Theorem 2.2, we see easily that every function in \( Q(\delta) \ (0 < \delta \leq 1) \) preserves the angles under the integral operator defined by (2.13).

By letting \( g(z) = z \) and \( B \to A \ (A < 1) \) in Theorem 2.2, we have

**Corollary.** If \( f \in A \) and

\[
|\arg \{(zf'(z))'\}| < \frac{\pi}{2} \delta \ (0 < \delta \leq 1 \ ; \ z \in U)
\]

then

\[
|\arg \{(zF'_\gamma(f))'\}| < \frac{\pi}{2} \eta
\]

where \( F_\gamma \) is given by (2.13) and \( \eta(0 < \eta \leq 1) \) is the solution of the equation:

\[
\delta = \eta + \frac{2}{\pi} \tan^{-1}\left( \frac{\eta}{1 + c} \right).
\]

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**References**


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